FUNDAMENTALS
OF
CALCULUS

CHHAGA
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by

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"The Calculus is the most interesting and, after arithmetic, the most useful branch of Mathematics."

"Considering how many fools can calculate, it is surprising that it should be thought either a difficult or a tedious task for any other fool to learn how to master the same tricks.

What one fool can do, another can."

S. P. Thomson

"The method of fluxious (the Calculus) is probably one of the greatest, most subtle and sublime discoveries of any age; it opens a new world to our view and extends our knowledge as it were, to infinity." *Charles Hutton.*

"Pure Mathematics do remedy and cure many defects in the wit and faculties of individuals ; for if the wit be dull they sharpen it ; if too wandering they fix it." *Francis Bacon.*

No part of human knowledge could lay claim to the ti of Science before it had passed through the stage of math matical demostration." *Leonardo Da Vin*

"The Science of Pure Mathematics, in its modern developments, may claim to be the most original creation of the human spirit."

A. N. Whitehead.

"Nature's great book is written in mathematical language".
Galileo

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CHAPTER I

FUNCTIONS AND THEIR LIMITS

"All things are in a state of change or flux ; nothing remains unaltered for an instant. It is a moving, changing universe with which we have to do." *Hugh Elliot.*

1.1. Differential Calculus. Change and motion is the rule of life. There is hardly anything in this world which does not change or is stationary. The differential calculus deals with study of variation, change and motion and the relationship between the changes of related variables. It is the most powerful mathematical tool that has ever been discovered. Our present knowledge of sound, light electricity and atomic energy could not have been discovered without the aid of calculus.

Functions and limits form the basic material of calculus and this first chapter is devoted mainly to the development of the fundamental concepts of function and limit.

In calculus we have to deal with two kinds of quantities : (1) *Constants* which retain the same value throughout an investigation and which are usually denoted by the letters such as $a, b, c, d, \ldots$ and (2) *Variables* which are free to take different values and are denoted by letters such as x, y, z, u, v, w, t, etc.

1 2 Function. *If two variables x and y are so related that, when the value of x is given the value of y is determined, then y is said to be a function of x.*

One variable x to which we assign values at pleasure, is called the *independent variable.* The other variable y, the value of which depends upon the value of x, is called the *dependent variable.*

Thus the area of a circle is function of its radius. If we denote the area by y and the radius by x, we have $y=\pi x^2$. This equation gives a relation between the independent variable x and the dependent variable y, π is here a constant.

Similarly the distance travelled by a train moving with a constant speed u, is a function of time. This is expressed by the relation $s=ut$, where t, the time, is the independent variable, s the distance, the dependent variable and u is a constant.

1.21. In general, if y is a function of x, then x is likewise a function of y, and either may be chosen as independent variable according to convenience.

Thus the equation $y=\pi x^2$ can also be written as $x=\sqrt{y/\pi}$ giving the radius x when the area y is known.

All expressions involving x such as, x^2+5x+7, $\sin x$, $\log x$, e^x are functions of x.

1.22. Functional notation. A particular function of x is often represented by the notation $f(x)$, which is read as function of x or *f of x* (not f times x).

If it is necessary to deal with several functions in the same discussion, they are distinguished either by the use of subscripts or by different letters. Thus $f(x)$, $f_1(x)$, $f_2(x)$, $F(x)$, $\phi(x)$, $g(x)$, represent different functions of x.

1.23. Value of a function. The value of a function $f(x)$ for any particular value a of x is indicated by replacing x by its value a. Thus $f(o)$, $f(3)$ and $f(a)$ denote the values, of $f(x)$ which correspond, respectively to $x=o$, $x=3$ and $x=a$

For example, if $f(x)=x^2+3x+1$, $f(o)=1$, $f(1)=5$,

$$f(-1)=-1,\ f(3)=19,\ f(y)=y^2+3y+1$$
$$f(x^2)=(x^2)^2+3(x^2)+1=x^4+3x^2+1.$$

Examples

Ex. 1. If $f(x)=2x^2+3x+7$, show that $12f(0)=7f(1)$.

We have, $f(0)=7$ and $f(1)=2+3+7=12$.

$\therefore\quad 12f(0)=12\times 7$ and $7f(1)=7\times 12$.

Hence $\quad 12f(0)=7f(1)$.

Ex. 2. If $(\theta)=\cos\theta+\sin\theta$, show that $f(0)=f(\frac{1}{2}\pi)$ and find $f(\frac{1}{4}\pi-\theta)$.

Here $\quad f(0)=\cos 0+\sin 0=1+0=1$

$$f(\tfrac{1}{2}\pi)=\cos\tfrac{1}{2}\pi+\sin\tfrac{1}{2}\pi=0+1=1$$

$\therefore\quad f(0)=f(\frac{1}{2}\pi)$.

Also $\quad f(\frac{1}{4}\pi-\theta)=\cos(\frac{1}{4}\pi-\theta)+\sin(\frac{1}{4}.\pi-\theta)$

$$=\frac{1}{\sqrt{2}}(\cos\theta+\sin\theta)+\frac{1}{\sqrt{2}}(\cos\theta-\sin\theta)$$
$$=\frac{2}{\sqrt{2}}\cos\theta.\ =\sqrt{2}\cos\theta.$$

Ex. 3. If $f(x)=\log x$, show that $f(ab)=f(a)+f(b)$.

Here $\quad f(a)=\log a$, $f(b)=\log b$ and $f(ab)=\log(ab)$

$\therefore\quad f(ab)=\log(ab)=\log a+\log b=f(a)+f(b)$.

Ex. 4. Find $f[\phi(3)]$ where $f(x)=2x-3$ and $\phi(x)=x^2+1$

We have $\phi(3)=3^2+1=10$

$\therefore\quad f[\phi(3)]=f(10)=2\times 10-3=17$.

Exercise 1-A

✓ **1.** If $f(x)=x^3+5x+6$, find the values of
$f(3)$, $f(0)$, $f(-1)$ and $f(-2)$.

✓ **2.** If $f(x)=3x^3+2x^2-x+4$, show that $2f(3)=25f(1)$.

✓ **3.** When $f(x)$ represents x^3-x^2+x-1. find $f(x+1)$ and $f(x-1)$.

✓ **4.** If $f(\theta)=\sin\theta-\cos\theta$, find

$$f(0),\quad f\left(\frac{\pi}{4}\right),\ f\left(\frac{\pi}{2}\right) \text{ and } f\left(\frac{3\pi}{4}\right).$$

Show that $$f\left(\frac{\pi}{3}\right)\times f\left(\frac{2\pi}{3}\right)=\tfrac{1}{2}.$$

5. Given $f(x)=\sin x-\cos x$, show that

$$f\left(\frac{\pi}{3}-x\right)+f\left(\frac{\pi}{6}+x\right)=0.$$

6. If $f(x)=\dfrac{1+\tan x}{1-\tan x}$, show that $f\left(\dfrac{\pi}{4}-x\right)=\cot x$.

7. Given $f(x)=\log x$, show that $f\left(\dfrac{1}{x}\right)=-f(x)$.

Exercise 1-B

Ex. 1. If $f(x)=\dfrac{2x-1}{x-2}=y$, show that $f(y)=x$.

Since $$\frac{2x-1}{x-2}=y$$

$\therefore$ $$2x-1=xy-2y$$

or $$2y-1=x(y-2)$$

i.e., $$x=\frac{2y-1}{y-2}=f(y).$$

2. If $y=\dfrac{x-1}{x+1}=f(x)$, express x as a function of y and show that $f\left(\dfrac{1}{y}\right)=\dfrac{1}{x}$.

3. If $f(x)=\dfrac{3x-5}{5x-3}=y$, show that $x=f(y)$.

4. If $f(x)=\dfrac{2x+1}{2x-1}$, show that $\dfrac{f(x)+1}{f(x)-1}=2x$.

1.3. Even and odd functions. A function of x is said to be an *even* function of x when $f(-x)=f(x)$, *i.e.*, when the change in the sign of x does not change the function in magnitude as well as in sign. It is said to be an *odd* function if $f(-x)=-f(x)$. *i.e.*, the function changes in sign only.

Exercise 1-C

1. Show that the following functions are even functions of x.

(*i*) $ax^6+bx^4+cx^2+d$. (*ii*) $\text{Cos } x+\text{Sec } x$.

(*iii*) e^x+e^{-x}. (*iv*) $\dfrac{b^x-b^{-x}}{3x+x^3}$.

2. Show that the following functions are odd.

(*i*) ax^5+bx^3+cx. (*ii*) $\text{Sin } \theta+\text{Cosec } \theta$.

(*iii*) $\dfrac{ax^3+bx}{x^2+1}$. (*iv*) $\dfrac{e^x+1}{e^x-1}$. (*v*) $\dfrac{b^x+b^{-x}}{3x+x^3}$.

1.4. It may be observed that in the case of certain functions of x, it is not possible to give all values to x. The range of values of x for which the function is defined must be indicated whether it is defined for all values of x or only for a limited number of values of x; *e.g.*, the function $x\,!$ (factorial x) has only a meaning when x is a positive integer.

Again, such a function as $\sqrt{1-x^2}$ is defined for all values of x which are numerically less than or equal to 1; in this case we may say that the function is defined for values of x in the range -1 to $+1$, both inclusive.

The function $\dfrac{1}{x-3}$ is defined for all values of x

except $x=3$, because for $x=3$, the function becomes 1 divided by zero, which is meaningless as division by zero is an impossible operation.

Note. The student can see for himself that if division by zero is allowed, it would lead to contradictory results, for example

$$0 \times 5 = 0 \times 7 \qquad \text{(both sides being equal to zero)}$$

Dividing both sides by 0,

we get, $5=7$, which is not correct.

Exercise 1-D

1. For what values of x are the following functions defined ?

(*i*) $\sqrt{x}$. (*ii*) $\sqrt{(x-1)(x-2)}$. (*iii*) $\text{Sin}^{-1} x$.

2. For what values of x are the following functions not defined ?

(*i*) $\dfrac{1}{x-2}$, (*ii*) $\dfrac{1}{(x-1)(x-2)}$, (*iii*) $\dfrac{1}{x^2-9}$,

(*iv*) $\tan x$, (*v*) $\text{Sec}^{-1} x$.

3. Point out the fallacy in the following :—

Let $x=3$, then $x^2=3x$ and $x^2-9=3x-9$

i.e., $(x-3)(x+3)=3(x-3)$ $\therefore$ $x+3=3$

Hence $6=3$ (Since $x=3$.)

Exercise 1-E

1. If $f(x)=\dfrac{x^2+x+1}{x}$, show that

(*i*) $f(a)=f\left(\dfrac{1}{a}\right)$; (*ii*) $f(b^2)=[f(b)]^2-2f(b)$.

2. If $f(x)=\log(x+1)$, show that

$$f(z^2-2)-f(z-2)=f(z).$$

3. If $f(\theta)=\tan\theta$, show that

$$f(\alpha+\beta)=\frac{f(\alpha)+f(\beta)}{1-f(\alpha)\,f(\beta)}.$$

4. If $y=x^2+5$, what is the increase in y when x increases from (*i*) 1 to 2, and (*ii*) 2 to 3 ?

5. If $f(x)=\frac{24}{x}$ what is the increase in $f(x)$ when x increases from (*i*) 2 to 3, and (*ii*) 3 to 4.

6. If $y=x^2$, what is the value of y for $x=10$? What is the change in the value of y when x is increased by 2 ? What is the ratio of the change of y to that of x ?

7. If $\phi(x)=x^2$, find $\phi(x+h)$, $\phi(x+h)-\phi(x)$ and

$$\frac{\phi(x+h)-\phi(x)}{h}.$$

Show that you obtain the same result as in example 6, when you put $x=10$ and $h=2$.

8. If $f(x)=x^2-3x+6$, find $f(3)$, $f(3+h)$ and

$$\frac{f(3+h)-f(3)}{h}.$$

The Limit of a Function

1.5. In order to have an idea of limits of functions we shall first take three preliminary notions :

(1) **The pharse "x tends to infinity."** ($x \to \infty$) Suppose we give to x the values 1, 2, 3, 4......successively, then it is not difficult to see that if this process continues x will get larger and larger values and there will be no end to the extent of this increase. However large a number we may think of (*e g.*, 10,000,000) x will have a larger value than this at some stage of its development. This unending growth of x is expressed by saying that *x tends to infinity* or $x \to \infty$. By this process we simply mean that x is supposed to take a series of values which go on increasing beyond limit.

There is no number ∞, and such an equation as

$$x=\infty$$

as it stands, is absolutely meaningless ; x cannot be 'equal to ∞.' because 'equal to infinity' means nothing.

(2) **The phrase 'x tends to zero.'** ($x \to 0$) In a similar manner let x take, say for the sake of definiteness, certain positive values, which continually decrease; like, for instance. $\frac{1}{10}$, $\frac{1}{10^2}$, $\frac{1}{10^3}$, $\frac{1}{10^4}$............Then, as before, if this process continues, a stage will come when x has become smaller than any quantity, however small $\left(e.g., \frac{1}{10,000,000}\right)$. If we think of a quantity still smaller than this, x will be smaller than that at a further stage, and this process will always continue. The unending decrease of x is expressed by saying that x *tends to zero* or $x \to 0$. This is different from saying that $x=0$ which means that x has actually taken the value zero ; while $x \to 0$ means that x is a variable taking values one after another which become smaller and smaller and which are such that the difference between x and zero becomes and remains less than any asigned positive number, however small. *The zero* (0) *is entirely different from a number, however small.*

(3) **The phrase 'x tends to a.'** ($x \to a$) Now let us have a finite number a, and we may imagine x to assume successively values say, $a+\frac{1}{10}$, $a+\frac{1}{10^2}$, $a+\frac{1}{10^3}$, $a+\frac{1}{10^4}$.........Then as x takes these values, it is obvious that the difference between the value of x and a continually decreases. At a certain stage this difference will become smallar than any quantity however small we may think of $\left(e.g., \frac{1}{10,000,000}\right)$. At a further stage the difference will be still smaller and this process will always continue. In this case we say that '*x tends to a*' or $x \to a$ and by this phrase we mean that x is taking values one after another which are

such that the difference between a and the successive values of x becomes and remains less than any assigned positive number, however small.

1.6. Consider now a function of x, *i.e.*, a variable whose value depends upon that of x and try to find out what value does $f(x)$ approach as x tends to a certain value. Let us take the function $f(x)=x^2$ and find its limit when x approaches the value 10. We make the following table for different values of x near $x=10$.

x	x^2	$10-x$	$100-x^2$
9·9	98·01	·1	1·99
9·99	99·8001	·01	0·1999
9·999	99·980001	·001	0·019999
...	...	...	...
...	...	...	...
...	...	...	...
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
10	100	0	0

From these figures it appears that as $x \to 10$, $x^2 \to 100$, and as the difference $10-x$ tends to 0, $100-x^2$ also tends to 0.

It is not necessary that x should approach 10 from values which are less than 10. We may just as well take values of x greater than 10.

Suppose

$x=10{\cdot}1$, then $x^2=102{\cdot}01$;
$x=10{\cdot}01$, ,, $x^2=100{\cdot}2001$;
$x=10{\cdot}001$, ,, $x^2=100{\cdot}020001$;

The values of x^2 again approach 100 as $x \to 10$.

The value of x^2 for $x=10$ is also 100. In this case, the value and limit happen to be equal, but this is not always the case.

1.61. Let us find the limit of the function $\frac{x^2-100}{x-10}$ as $x \to 10$.

x	$\frac{x^2-100}{x-10}$	$10-x$	$20-\frac{x^2-100}{x-10}$
9·9	$\frac{98\cdot01-100}{9\cdot9-10}=19\cdot9$	·1	·1
9·99	$\frac{99\cdot8001-100}{9\cdot99-10}=19\cdot99$	·01	·01
9·999	$\frac{99\cdot980001-100}{9\cdot999-10}=19\cdot999$	·001	·001
... ... ↓ 10	... ... ↓ 20	... ... ↓ 0	... ... ↓ 0

In this case it may be observed that

(*i*) When $x=10$, the value of the expression $\frac{x^2-100}{x-10}$ $=\frac{0}{0}$, which has no meaning since the operation of dividing any number by 0 is meaningless. When an expression takes the form $\frac{0}{0}$, it is said to be *indeterminate.*

(*ii*) As $x \to 10$, $\frac{x^2-100}{x-10} \to 20$.

(*iii*) In this case the limit and value are not equal. In fact the value does not even exist while the limit is 20.

1.62. Generally we say that $f(x)$ tends to a definite limit l as x tends to a if the difference between $f(x)$ and l can be made as small as we please by making x approach sufficiently near a and write

$$\underset{x \to a}{\text{Lt}}\ f(x) = l.$$

The limit of $f(x)$ as $x \to a$, does not mean the value of $f(x)$ when $x=a$. To find the value of $f(x)$ for $x=a$ we merely substitute $x=a$ in $f(x)$ and obtain $f(a)$. But in taking limit of $f(x)$ as $x \to a$, we find what value does $f(x)$ continually approach as x gets nearer and nearer a, or in other words, what $f(x)$ is very nearly equal to when x is very nearly equal to a. This limit of $f(x)$ as $x \to a$ is not necessarily equal to the value $f(a)$.

1.7. Continuous and Discontinuous functions. If, however, the limit of $f(x)$ as x approaches a is equal to its value $f(a)$; then $f(x)$ is said to be continuous at $x=a$. It is clear from Art. 1.6 that x^2 is continuous at $x=10$, because both the value and limit are equal to 100. As a matter of fact, most of the ordinary functions will be found to be continuous at all points of their range of definition.

The definition of continuity of $f(x)$ at $x=a$ involves three conditions :—

(*i*) $f(a)$ should exist

(*ii*) $\underset{x \to a}{\text{Limit}}\ f(x)$ should exist

(*iii*) $f(a) = \underset{x \to a}{\text{Lt}}\ f(x)$

It is easy to construct examples in which one or more of the above conditions is not fulfilled. In that case the function $f(x)$ is not continuous at $x=a$ A function which is not continuous at $x=a$ is said to be discontinuous at $x=a$. Art. 1.61 gives an example of a discontinuous function. $f(x) = \dfrac{x^2-100}{x-10}$ is discontinuous at $x=10$, because in this case $f(10)$ does not exist.

If $f(x)$ is continuous at all points between a and b including both a and b, then $f(x)$ is said to be continuous in the interval (a, b). For further examples see Art. 5·2.

1.8. Theorems on limits.

If $\underset{x\to a}{\text{Lt}}\, f(x)=m$ and $\underset{x\to a}{\text{Lt}}\, \phi(x)=n$;

then (*i*) $\underset{x\to a}{\text{Lt}}\, [f(x)\pm\phi(x)]=m\pm n$. (*ii*) $\underset{x\to a}{\text{Lt}}\, f(x)\phi(x)=mn$.

(*iii*) $\underset{x\to a}{\text{Lt}}\, \dfrac{f(x)}{\phi(x)}=\dfrac{m}{n}$, provided $n\neq 0$.

These theorems may be taken for granted. A formal proof of these is beyond the scope of the present work.

Examples

1.81. Ex. 1. If $f(x)=\dfrac{1}{x}$, find the limit of $f(x)$ as $x\to\infty$.

x	10	100	1000	10000	...	$\to\infty$
$\dfrac{1}{x}$	·1	·01	·001	·0001	...	$\to 0$

As x gets larger and larger, $\dfrac{1}{x}$ becomes smaller and smaller. When x becomes very large, $\dfrac{1}{x}$ becomes very small and thus as $x\to\infty$, $\dfrac{1}{x}\to 0$, so that

$$\underset{x\to\infty}{\text{Lt}}\, \frac{1}{x}=0.$$

It can be shown similarly that as $x\to\infty$, $x^p\to\infty$ if p is positive and $x^p\to 0$ if p is negative.

Ex. 2. If $f(n)$, a function of n is defined by the series :

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\dots\dots\dots+\frac{1}{2^n},$$

find the limit of $f(n)$ as $n \to \infty$, where n is a positive integer.

$f(n)$=sum of n terms of the given geometric series

$$=\frac{\frac{1}{2}\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}}=1-\frac{1}{2^n}.$$

Now as $n \to \infty$, $2^n \to \infty$ and therefore $\frac{1}{2^n} \to 0$.

Hence $f(n) \to 1$ as $n \to \infty$.

Exercise II-A

1. Show by taking different values of x that $\frac{1}{x^2}$ becomes smaller and smaller as x becomes larger and larger.

2. Find the limit of $f(n)$ when $n \to \infty$, n being a positive integer, where

(a) $f(n)=1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots\ldots+\frac{1}{3^n}$

(b) $f(n)=-1+\frac{1}{3}-\frac{1}{3^2}+\ldots\ldots+\frac{1}{3^{2n-1}}$

(c) $f(n)=1-\frac{1}{5}+\frac{1}{5^2}-\frac{1}{5^3}+\ldots.+\frac{1}{5^{2n}}$.

1.82. Ex. 1. Find the limit of $\frac{x^2-4}{x-2}$ as $x \to 2$.

Here if we put $x=2$, the expression takes the form $\frac{0}{0}$, which is meaningless. To find the limit, we put $x=2+h$, where h is a small quantity approaching zero.

Hence $\underset{x \to 2}{\mathrm{Lt}} \frac{x^2-4}{x-2}=\underset{h \to 0}{\mathrm{Lt}} \frac{(2+h)^2-4}{2+h-2}$

$$=\mathrm{Lt}\frac{4+4h+h^2-4}{h}=\mathrm{Lt}(4+h).$$

Now as $x \to 2$, $h \to 0$ $\therefore$ $4+h \to 4$

i.e.,
$$\underset{x-2}{\text{Lt}} \frac{x^2-4}{x-2}=4.$$

Aliter.
$$\frac{x^4-4}{x-2}=\frac{x-2}{x-2}(x+2).$$

Now as x is given a series of values approaching 2 but not actually equal to 2, $x-2$ is not zero and therefore division by $x-2$ is valid.

Hence the given expression $=x+2=4$, when $x=2$.

Note.—This is an example of a function where the numerator and denominator both approach the limit zero but the function itself approaches a finite limit.

Ex. 2. Find the limit of $\dfrac{\sin\theta-\cos\theta}{\tan\theta-\cot\theta}$

when θ approaches the limit $\dfrac{\pi}{4}$.

$$\frac{\sin\theta-\cos\theta}{\tan\theta-\cot\theta}=\frac{\sin\theta-\cos\theta}{\dfrac{\sin^2\theta-\cos^2\theta}{\sin\theta\,.\,\cos\theta}}$$

$$=\frac{\sin\theta-\cos\theta}{\sin^2\theta-\cos^2\theta}\,.\,\sin\theta\cos\theta.$$

Now so long as θ is not actually equal to $\dfrac{\pi}{4}$,

$\sin\theta-\cos\theta$ is not zero and, therefore, division by $\sin\theta-\cos\theta$ is permissible.

Hence the limit of the given expression

$$=\text{Lt}\frac{\sin\theta\,.\,\cos\theta}{\sin\theta+\cos\theta}=\frac{1}{2\sqrt{2}}, \text{ when } \theta \to \frac{\pi}{4}$$

Ex. 3. Evaluate $\underset{x\to\infty}{\text{Lt}} \dfrac{x^2+5x+6}{3x^2+7x+11}$.

This expression takes the form $\dfrac{\infty}{\infty}$ which is *indeterminate.*

Dividing the numerator and the denominator by x^2 (the highest power of x in the denominator) the expression becomes.

$$\frac{1+\dfrac{5}{x}+\dfrac{6}{x^2}}{3+\dfrac{7}{x}+\dfrac{11}{x^2}}.$$

Now as $x\rightarrow\infty$, $\dfrac{1}{x}\rightarrow 0$ and $\dfrac{1}{x^2}\rightarrow 0$.

Hence the limiting value of the expression as $x\rightarrow\infty$ becomes $\frac{1}{3}$.

Exercise II-B

1. Calculate the limiting value of $\dfrac{x^2-3x+2}{x^2-4x+3}$ when $x\rightarrow 1$.

2. Find the value of $\dfrac{x^2-5x+6}{x^2-8x+15}$ as $x\rightarrow 3$.

3. Find the value of $\underset{x\rightarrow 0}{\text{Lt}}\dfrac{2x^2+3x^3+x^4}{3x^2+x^3+3x^4}$.

4. Evaluate :

(*i*) $\underset{x\rightarrow 1}{\text{Lt}}\dfrac{x^2-1}{x-1}$; (*ii*) $\underset{x\rightarrow a}{\text{Lt}}\dfrac{x^2-a^2}{x-a}$; (*iii*) $\underset{x\rightarrow h}{\text{Lt}}\dfrac{x^3-h^3}{x-h}$.

5. Find $\text{Lt}\dfrac{ax+b}{bx+a}$ (*i*) when $x\rightarrow 0$. (*ii*) when $x\rightarrow\infty$.

6. Find the limit when $x\rightarrow\infty$ of :

(*i*) $\dfrac{x+5}{x+3}$; (*ii*) $\dfrac{x^3+3x+5}{x^2+2x+3}$; (*iii*) $\dfrac{x^2+4x+5}{x^3+5x+6}$.

7. Given that $\underset{x\rightarrow 0}{\text{Lt}}\dfrac{\sin x}{x}=1$, deduce

(*i*) $\underset{\theta\rightarrow 0}{\text{Lt}}\dfrac{\sin a\theta}{\sin b\theta}=\dfrac{a}{b}$; (*ii*) $\underset{x\rightarrow 0}{\text{Lt}}\dfrac{\tan x}{x}=1$.

(*iii*) $\underset{x\rightarrow 0}{\text{Lt}}\dfrac{1-\cos x}{x}=0$.

8. Show that

(*i*) $\underset{x\to 0}{\text{Lt}}\ \dfrac{\sin x}{\sin 2x}=\frac{1}{2}$. (*ii*) $\underset{x\to\frac{\pi}{2}}{\text{Lt}}\ \dfrac{\tan 3x}{\tan x}=\frac{1}{3}$.

9. Show that

(*i*) $\underset{\theta\to 0}{\text{Lt}}\ \dfrac{\sin\theta-\tan\theta}{\tan^2\theta}=0$. (*ii*) $\underset{\theta\to 0}{\text{Lt}}\ \dfrac{\sin\theta-\tan\theta}{\tan^3\theta}=-\frac{1}{2}$.

(*iii*) $\underset{x\to\pi/2}{\text{Lt}}\ \dfrac{\cot x-\cos x}{\cos^3 x}=\frac{1}{2}$.

1.83. Ex. 1. Find the limit of $\sqrt{x+1}-\sqrt{x}$ as $x\to\infty$.

This expression is of the form $\infty-\infty$ which is also one of the *indeterminate* forms like $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

To find the limit we multiply and divide the given expression by $\sqrt{x+1}+\sqrt{x}$, we get

$$\sqrt{x+1}-\sqrt{x}=\frac{(\sqrt{x+1}-\sqrt{x})(\sqrt{x+1}+\sqrt{x})}{\sqrt{x+1}+\sqrt{x}}.$$

$$=\frac{1}{\sqrt{x+1}+\sqrt{x}}$$

Now as $x\to\infty$, the denominator $\to\infty$ and the whole fraction $\to 0$.

Henec $\underset{x\to\infty}{\text{Lt}}\sqrt{x+1}-\sqrt{x}=0$.

Ex. 2. Find the limit of $z-\sqrt{(z+2)(z-3)}$ when $z\to\infty$.

Here $z-\sqrt{(z+2)(z-3)}$

$$=\frac{[z-\sqrt{(z+2)(z-3)}][z+\sqrt{(z+2)(z-3)}]}{z+\sqrt{(z+2)(z-3)}}$$

$$=\frac{z^2-(z^2-z-6)}{z+\sqrt{z^2-z-6}}=\frac{z+6}{z+\sqrt{z^2-z-6}}$$

Dividing the numerator and the denominator by z,

we have, the expression $=\dfrac{1+\dfrac{6}{z}}{1+[1-\dfrac{1}{z}-\dfrac{6}{z^2}]^{\frac{1}{2}}}$

Now as $z\to\infty$, $\dfrac{1}{z}$, $\dfrac{1}{z^2}\to 0$.

Hence the limit$=\dfrac{1}{1+1}=\frac{1}{2}$.

Exercise II-C

1. Prove that :

(*i*) $\underset{x\to 0}{\text{Lt}}\ \dfrac{x}{1-\sqrt{1-x}}=2.$

(*ii*) $\underset{x\to 0}{\text{Lt}}\ \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}=1.$ (*iii*) $\underset{x\to 0}{\text{Lt}}\ \dfrac{(1-x)^2-1}{x}=-2.$

2. Find the limit of :

(*i*) $x-\sqrt{x(x-1)}$. (*ii*) $x-\sqrt{(x+1)(x+2)}$ when $x\to\infty$.

1.84. Ex. If $f(x)=x^3$, find the limit of

$$\frac{f(x+h)-f(x)}{h} \text{ as } h\to 0.$$

Since $$f(x)=x^3$$
$$f(x+h)=(x+h)^3$$

and $$f(x+h)-f(x)=(x+h)^3-x^3=3x^2h+3xh^2+h^3$$

$\therefore$ $$\frac{f(x+h)-f(x)}{h}=\frac{3x^2h+3xh^2+h^3}{h}$$
$$=3x^2+3xh+h^2.$$

Hence $$\underset{h\to 0}{\text{Lt}}\ \frac{f(x+h)-f(x)}{h}=3x^2.$$

Exercise II-D

1. If $f(x)=5x-x^2$, express $\dfrac{f(1+h)-f(1)}{h}$ in terms of h and find the limit when $h\to 0$.

2. Given $f(x)=x^2-x+10$, find the limit of $\dfrac{f(2+h)-f(2)}{h}$ as $h\to 0$.

3. Given $f(x)=x^2$, find the limit of $\dfrac{f(x+h)-f(x)}{h}$ as $h\to 0$.

CHAPTER II

DIFFERENTIATION OF FUNCTIONS

'Observe always that everything is the result of a change and get used to thinking that there is nothing nature loves so much as to change existing forms and to make new ones like them.'

Differential Co-efficient

2·1. We have seen that change in a function depends upon a change in the independent variable. If $f(x)$ is a function of x, and x is increased by a quantity h, positive or negative, the function will become $f(x+h)$ and the increase in the function will be $f(x+h)-f(x)$. The ratio of the increment of the function to the increment of x is $\dfrac{f(x+h)-f(x)}{h}$.

This ratio is called the *Quotient of Differences*.

Now it is evident that as h, the increase in x, becomes smaller and smaller $f(x+h)-f(x)$, the change in the value of the function, also becomes smaller and smaller. Ultimately, when $h \to 0$, $f(x+h)-f(x)$ also $\to 0$. We have already dealt with certain fractions whose numerator and denominator both approach the limit zero and the fraction itself tends to a finite limit. The present problem is also similar and it can be shown by means of examples that in general the limit of

$\dfrac{f(x+h)-f(x)}{h}$ as $h \to 0$, is finite.

Our chief aim in Differential Calculus is to find the limit of this fraction and this limit is called the *Differential Co-efficient* or the *Derivative* of the function $f(x)$.

2·2. Definition.

The Differential Co-efficient or the derivative of a function is, therefore, defined as the limit (when it *exists) of* $\dfrac{f(x+h)-f(x}{h}$ *when the denominator h (which may be positive or negative) becomes very small and approaches the limit zero.*

2.21. *Illustration.*

Suppose $f(x)=x^2$. When $x=3, f(x)=9$. If certain increments are given to x, $f(x)$ also gets the corresponding increments. The following table shows the relation between these increments :—

Increment of x	Increment of $f(x)$	Quotient of differences
h	$f(x+h)-f(x)$	$\dfrac{f(x+h)-f(x)}{h}$
h	$(3+h)^2-9=6h+h^2$	$6+h$
—·1	—·59	5·9
—·01	—·0599	5·99
—·001	—·005999	5·999
—·0001	—·00059999	5·9999
...	...	...
...	...	...
+·1	+·61	6·1
·01	·0601	6·01
·001	·006001	6·001
·0001	·0006001	6·0001
...	...	...
...	..	...
↓	↓	↓
0	0	6

It is evident from this table that as h becomes smaller and smaller in magnitude, $f(x+h)-f(x)$ also becomes smaller and smaller but the ratio of the two increments or quotient of differences approaches the value 6. Thus for the value $x=3$, the differential coefficient of x^2 is 6.

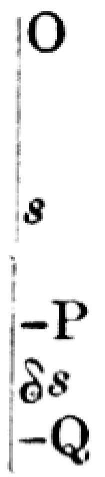
2.22. The function $f(x)$ is generally put equal to another variable y and we usually denote the increment in x by the symbol δx (delta x) and the increment in the function by $\delta[f(x)]$ or δy (delta y). Thus δx corresponds to h and δy stands for $f(x+h)-f(x)$. The quotient of differences is then $\dfrac{\delta y}{\delta x}$. The limit of $\dfrac{\delta y}{\delta x}$ as δx approaches zero is the differential co-efficient of y with respect to x and is denoted by the symbol $\dfrac{dy}{dx}$ or $\dfrac{d}{dx}[f(x)]$.

Similarly if y is a function of some other independent variable t, its derivative is denoted by $\dfrac{dy}{dt}$.

Note. The symbol $\dfrac{d}{dx}$ stands for an operation which when applied to y, is the operation of taking the limit of $\dfrac{\delta y}{\delta x}$ when $\delta x \to 0$, *i.e.*, when $\dfrac{d}{dx}$ is prefixed to an expression, it means that the expression is to be differentiated with respect to x and it does not mean $d \div dx$.

The symbol $\dfrac{d}{dx}$ may be compared with the symbol $\surd$ which stands for the operation of extracting the square root. $\sqrt{x}$ does not mean $\surd$ multiplied by x.

2·23. *Illustration.*



Suppose the distance OP described by a body falling vertically due to the attraction of the earth is given by the equation $s=10t+16t^2$.

s, the space described in time $t = \text{OP} = 10t+16t^2$

$s+\delta s$, space described in time $t+\delta t$

$$=\text{OQ}=10(t+\delta t)+16(t+\delta t)^2$$

δs, space described in time δt,

$$=\text{PQ}=\text{OQ}-\text{OP}=10\delta t+32t.\delta t+16(\delta t)^2$$

$\frac{\delta s}{\delta t}$, average speed in the interval $\delta t = \frac{PQ}{\delta t}$

$$= 10 + 32t + 16\delta t.$$

Hence $\frac{ds}{dt}$, the speed at the time t

$$= \underset{\delta t \to 0}{\text{Lt}} \frac{PQ}{\delta t} = 10 + 32t.$$

Differentiation

2.3. *Differentiation of a function is the process of finding the differential co-efficient of the function.* The definition of the differential co-efficient leads to the following process of differentiating y or $f(x)$:—

(*i*) *Give to x the increment δx which may be positive or negative and obtain*

$$y + \delta y \text{ or } f(x + \delta x).$$

(*ii*) *Find δy, the increment of the function or* $f(x + \delta x) - f(x)$.

(*iii*) *Obtain the quotient of differences or*

$$\frac{\delta y}{\delta x} \text{ or } \frac{f(x + \delta y) - f(x)}{\delta x}.$$

(*iv*) *Proceed to find the limit of this quotient as δx approaches zero.*

The result so obtained is the required differential co-efficient of y or $f(x)$ which is written symbolically in the form

$$\frac{dy}{dx} \quad \text{or} \quad - \quad f(x)\,\Big].$$

Examples

Ex. 1. Find the differential co-efficient of x^2.

Let $\qquad y = x^2 \qquad \ldots(i)$

When δx is the increment given to x, let δy be the corresponding increment of y ; *i.e.*, when x becomes $x + \delta x$, let y become $y + \delta y$, then

$$y + \delta y = (x + \delta x)^2 \qquad \ldots(ii)$$

Subtracting (*i*) from (*ii*), we have

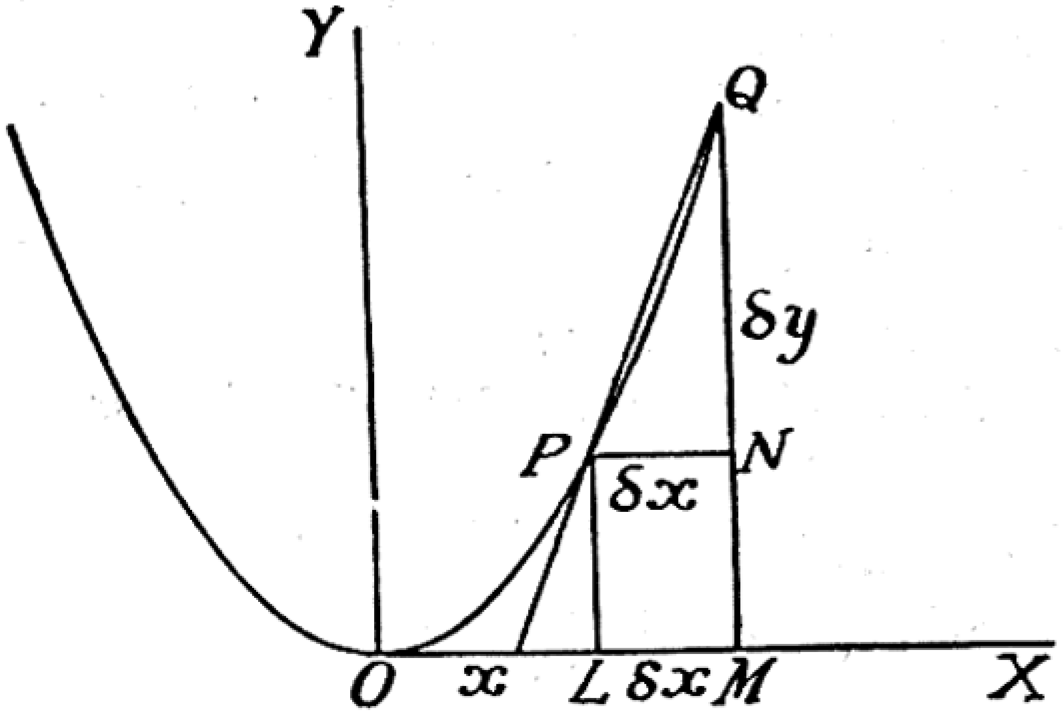
$$\begin{aligned}\delta y &= (x+\delta x)^2 - x^2\\ &= 2x.\delta x + (\delta x)^2.\end{aligned}$$

Dividing both sides by δx, we get

$$\frac{\delta y}{\delta x} = 2x + \delta x.$$

Now proceeding to the limit as $\delta x \to 0$, we have

$$\frac{dy}{dx} = 2x.$$



Thus the differential co-efficient of x^2 is $2x$ which becomes equal to 6 for $x=3$. It agrees with the result of illustration, Art. 2.21.

Note. Whether the increment δx is positive or negative, the final result will be the same in both cases.

Ex. 2. Let $\quad f(x) = 5x^3 + 7 \quad$ ...(*i*)

If δx is the increment given to x, then

$$f(x+\delta x) = 5(x+\delta x)^3 + 7 \qquad \text{...(}ii\text{)}$$

Subtracting (*i*) from (*ii*),

$$\begin{aligned}f(x+\delta x) - f(x) &= 5(x+\delta x)^3 - 5x^3\\ &= 5[3x^2.(\delta x) + 3x\,(\delta x)^2 + (\delta x)^3].\end{aligned}$$

$$\frac{f(x+\delta x) - f(x)}{\delta x} = 5[3x^2 + 3x.\delta x + (\delta x)^2]$$

Now in the limit when $\delta x \to 0$, we obtain

$$\frac{d}{dx}[f(x)] = \underset{\delta x \to 0}{\text{Lt}} \frac{f(x+\delta x)-f(x)}{\delta x}$$

$$=5.3x^2=15x^2.$$

Thus $15x^2$ is the diff. co-eff. of $5x^3+7$.

Note. The additive constant 7 disappears; after the second step and the multiplying constant 5 remains the same throughout.

Ex. 3. Differentiate $\dfrac{1}{t}$ with regard to t.

Let $$y=\frac{1}{t} \quad \text{...}(i)$$

then $$y+\delta y=\frac{1}{t+\delta t} \quad \text{...}(ii)$$

$$\delta y=\frac{1}{t+\delta t}-\frac{1}{t}$$

$$\therefore \quad \frac{\delta y}{\delta t}=\frac{1}{\delta t}\left[\frac{1}{t+\delta t}-\frac{1}{t}\right]$$

$$=-\frac{1}{t(t+\delta t)}$$

In the limit as $\delta t \to 0$, we have,

$$\frac{dy}{dt}=-\frac{1}{t^2}$$

i.e., $$\frac{d}{dt}\left(\frac{1}{t}\right)=-\frac{1}{t^2}.$$

Ex. 4. Find the derivative of $\sqrt{x}$.

Let $$y=\sqrt{x} \quad \text{...}(i)$$

Then $$y+\delta y=\sqrt{x+\delta x}$$

$$\delta y=\sqrt{x+\delta x}-\sqrt{x} \quad \text{...}(ii)$$

$$\therefore \quad \frac{\delta y}{\delta x}=\frac{\sqrt{x+\delta x}-\sqrt{x}}{\delta x}$$

$$=\frac{(\sqrt{x+\delta x}-\sqrt{x})(\sqrt{x+\delta x}+\sqrt{x})}{\delta x(\sqrt{x+\delta x}+\sqrt{x})}$$

$$=\frac{1}{\sqrt{x+\delta x}+\sqrt{x}},$$

In the limit when $\delta x \to 0$ we have

$$\frac{dy}{dx}=\frac{1}{\sqrt{x}+\sqrt{x}}=\frac{1}{2\sqrt{x}},$$

i.e., $$\frac{d}{dx}(\sqrt{x})=\frac{1}{2\sqrt{x}}.$$

Ex. 5. Find, *from the definition*, the derivative of

$$\frac{x^2-ax+b}{x-a}.$$

Let $$y=\frac{x^2-ax+b}{x-a}=x+\frac{b}{x-a}.$$

then $$y+\delta y=x+\delta x+\frac{b}{x+\delta x-a}$$

and $$\delta y=\delta x+b\left[\frac{1}{x-a+\delta x}-\frac{1}{x-a}\right]$$

$$=\delta x-b\,\frac{\delta x}{(x-a+\delta x)(x-a)}$$

$$\therefore \quad \frac{\delta y}{\delta x}=1-\frac{b}{(x-a-\delta x)(x-a)}.$$

In the limit when $\delta x \to 0$, we have

$$\frac{dy}{dx}=1-\frac{b}{(x-a)^2}.$$

Exercise III

1. Define differential co-efficient of a function and from the definition find the differential co-efficients of :—

(*i*) x^3. (*ii*) $9x+8$. (*iii*) ax^2+b. (*v*) $\frac{4}{3}\pi x^3$.
(*v*) πr^2. (*vi*) $(5-x)(x-1)$. (*vii*) $ut+\frac{1}{2}gt^2$.

2. Differentiate from first principles :—

(*i*) $\frac{3}{x}$. (*ii*) $\frac{c}{x^2}$. (*iii*) az^{-1}. (*iv*) $\frac{1}{z}+\frac{1}{z^2}$.

(*v*) $\dfrac{x^2+1}{x}$. (*vi*) $\dfrac{x^2+2x+3}{x+2}$. (*vii*) $\sqrt{5z}$.

(*viii*) $x+\dfrac{1}{\sqrt{x}}$. (*ix*) $\sqrt{x+c}$. (*x*) $\sqrt{x^2+c^2}$.

3. Differentiate *ab initio* :—

(*i*) $\dfrac{1}{x+7}$. (*ii*) $\dfrac{1}{5-2x}$. (*iii*) $(x+5)^2$.

Calculate these derivatives for the particular values (*i*) $x=0$, (*ii*) $x=4$.

4. Find from the definition the differential co-efficients of :—

(*i*) $\sqrt{x}$. (*P. U.*)

(*ii*) $\dfrac{1}{\sqrt{x}}$. (*P. U.*)

(*iii*) $a+bx+cx^2$. (*P. U.*)

5. Differentiate from first principles :—

$\sqrt{1+x^2}$. (*D. U. 1934*)

6. A stone thrown up into the air rises s feet in t seconds where $s=50t-16t^2$. Find the velocity $\dfrac{ds}{dt}$.

What is its value when $t=1$ and $t=2$?

7. If $s=4+18t-3t^2$, find the velocity $\dfrac{ds}{dt}$ in terms of t. Find the velocity when $t=2$, 4 sec., and the time when the velocity is zero.

2·4. Differentiation of x^n, when n is any rational number.

Let $$y=x^n. \qquad \text{...(1)}$$

When δx is the increment given to x, let δy be the corresponding increment of y, so that

$$y+\delta y=(x+\delta x)^n \qquad \text{.. (2)}$$

Subtracting (1) from (2), we have

$$\delta y=(x+\delta x)^n-x^n=x^n\left[1+\frac{\delta x}{x}\right]^n-x^n$$

$$=x^n\left[1+n\,\frac{\delta x}{x}+\frac{n(n-1)}{2\,!}\left(\frac{\delta x}{x}\right)^2+\ldots\ldots\right]-x^n$$

$$=nx^{n-1}\,\delta x+\frac{n(n-1)}{2\,!}x^{n-2}(\delta x)^2+\ldots\ldots$$

Dividing by δx, we get

$$\frac{\delta y}{\delta y}=nx^{n-1}+\frac{n(n-1)}{2\,!}x^{n-2}\delta x+\text{[terms containing higher powers of }\delta x]$$

Hence $\dfrac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\dfrac{\delta y}{\delta x}=nx^{n-1}.$

***2·41.** The proof given above involves the use of infinite Binomial series, the use of which at this stage is undesirable. A complete proof in three stages is therefore, given below :

Case 1. When n is a positive integer.

Let
$$y=x^n$$
$$y+\delta y=(x+\delta x)^n$$
$$\delta y=(x+\delta x)^n-x^n$$
$$=nx^{n-1}.\delta x+\frac{n(n-1)}{2!}x^{n-2}.(\delta x)^2+\ldots\ldots(\delta x)^n\ (n\text{ terms})$$

$$\therefore\quad \frac{\delta y}{\delta x}=nx^{n-1}+\frac{n(n-1)}{2\,!}x^{n-2}.(\delta x)+\ldots\ldots+(\delta x)^{n-1}.$$

In the limit when $\delta x\to 0$, we have

$\dfrac{dy}{dx}=nx^{n-1}$, since all terms except the first tend to 0 and the number of terms is finite.

$$\therefore\quad \frac{d}{dx}(x^n)=nx^{n-1}.$$

Case 2. When n is a negative integer and equal to, say, $-m$.

Then $$y=x^n=x^{-m}=\frac{1}{x^m},\quad y+\delta y=\frac{1}{(x+\delta x)^m}$$

and $$\delta y=\frac{1}{(x+\delta x)^m}-\frac{1}{x^m}=\frac{x^m-(x+\delta x)^m}{x^m(x+\delta x)^m}$$

$$=-\frac{mx^{m-1}\delta x+\frac{m(m-1)}{2\,!}x^{m-2}(\delta x)^2+\ldots+(\delta x)^m}{x^m(x+\delta x)^m}$$

$$\therefore\quad \frac{\delta y}{\delta x}=-\frac{mx^{m-1}+\text{ terms involving }\delta x}{x^m(x+\delta x)^m}$$

$\therefore$ taking limits when $\delta x\to 0$, we have

$$\frac{dy}{dx}=-\frac{mx^{m-1}}{x^m.x^m}=-mx^{-m-1}.$$

Replacing $-m$ by n, it gives us

$$\frac{dy}{dx}=nx^{n-1}.$$

***Case 3.** When n is a fraction and equal, say, to $\frac{p}{q}$, where p and q are integers.

Then $y=x^n=x^{\frac{p}{q}}$, $y+\delta y=(x+\delta x)^{\frac{p}{q}}$

$$\therefore\quad \frac{\delta y}{\delta x}=\frac{(x+\delta x)^{\frac{p}{q}}-x^{\frac{p}{q}}}{\delta x}$$

Now put $x^{\frac{1}{q}}=z$, then $x=z^q$, $\quad x+\delta x=(z+\delta z)^q$

and $x^{\frac{p}{q}}=z^p$, $(x+\delta x)^{\frac{p}{q}}=(z+\delta z)^p$

$$\therefore\quad \frac{\delta y}{\delta x}=\frac{(z+\delta z)^p-z^p}{(z+\delta z)^q-z^q}=\frac{(z+\delta z)^p-z^p}{\delta z}\times\frac{1}{\frac{(z+\delta z)^q-z^q}{\delta z}}.$$

For an easier Proof, see Art. 3.6.

Now proceeding to the limit when $\delta x \to 0$ and $\delta z \to 0$ we have.

$$\frac{dy}{dx}=\underset{\delta z\to 0}{\text{Lt}}\frac{(z+\delta z)^p-z^p}{\delta z}\times \text{Lt}\frac{1}{\dfrac{(z+\delta z)^q-z^q}{\delta z}}$$

$$=\frac{pz^{p-1}}{qz^{q-1}}=\frac{p}{q}\,z^{p-q}=\frac{p}{q}z^{q\left(\frac{p}{q}-1\right)}$$

Replacing z^q by x and $\frac{p}{q}$ by n, we have

$$\frac{dy}{dx}=\frac{p}{q}\,x^{\frac{p}{q}-1}=nx^{n-1}.$$

Thus for all rational values of n, we have

$$\frac{d}{dx}(x^n)=nx^{n-1}.$$

Cor. (*i*) $\frac{d}{dx}(\sqrt{x})=\frac{d}{dx}\left(x^{\frac{1}{2}}\right)=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}$.

(*ii*) $\frac{d}{dx}\left(\frac{1}{x}\right)=\frac{d}{dx}\left(x^{-1}\right)=-1x^{-2}=-\frac{1}{x^2}$.

(*iii*) $\frac{dx}{dx}=\frac{d}{dx}(x)=1.x^{1-1}=x^0=1.$

Similarly $\frac{du}{du}=1$, and $\frac{dz}{dz}=1.$

i. e., the differential co-efficient of a variable with respect to itself is always unity.

2.5. Differentiation of $(ax+b)^n$ when n is a positive integer.

Let $y=(ax+b)^n$

$$y+\delta y=[a(x+\delta x)+b]^n=[ax+b+a\delta x]^n$$

$$\delta y=[ax+b+a\delta x]^n-(ax+b)^n$$

$$=(ax+b)^n+n(ax+b)^{n-1}.a\delta x+\frac{n(n-1)}{2\,!}\times (ax+b)^{n-2}a^2(\delta x)^2+\ldots\ldots+a^n(\delta x)^n-(ax+b)^n$$

$\therefore\ \dfrac{\delta y}{\delta x}=an(ax+b)^{n-1}+(n-1)$ terms involving δx and powers of δx.

In the limit when $\delta x \to 0$, we have

$$\frac{dy}{dx}=an(ax+b)^{n-1}$$

since the number of terms is finite and all terms excepting the first vanish with δx, in the limit.

The result holds true for all rational values of n, the other two cases can be proved in the manner indicated in the last article. For another proof, see Ex. 1, Art. 3·6.

Examples

Ex. 1. If $y=(3x+2)^5$, $\dfrac{dy}{dx}=3\times 5(3x+2)^4=15(3x+2)^4$.

Ex. 2. If $y=\sqrt{5-7x}=(5-7x)^{\frac{1}{2}}$

$$\frac{dy}{dx}=(-7).\tfrac{1}{2}(5-7x)^{\frac{1}{2}-1}=\frac{-7}{2\sqrt{5-7x}}.$$

Exercise IV.

1. Write down the derivatives of :—

(*i*) x^9. (*ii*) $7x^3$. (*iii*) x^0. (*iv*) $\dfrac{3x^2}{4}$. (*v*) $0{\cdot}7x$. (*vi*) $0{\cdot}3x^{10}$

(*vii*) $3t^2$. (*viii*) $\dfrac{1}{2\sqrt{x}}$. (*ix*) $\dfrac{z^4}{4}$. (*x*) $\dfrac{1}{t}$

(*xi*) $3x^{-9}$. (*xii*) $\dfrac{1}{\theta^{\frac{3}{2}}}$. (*xiii*) $7x^{\frac{1}{7}}$. (*xiv*) $\dfrac{3}{8x^5}$.

(*xv*) $\sqrt[3]{8x^3}$. (*xvi*) $\sqrt[19]{u^3}$

2. If $y=x^5$, prove that $x\dfrac{dy}{dx}=5y$.

3. If $s=kv^{\frac{3}{8}}$; find $\frac{ds}{dv}$.

4. If the volume of a sphere is given by the formula $v=\frac{4}{3}\pi r^3$, find $\frac{dv}{dr}$ in terms of r and its value when $r=3$.

Also find the value of r for which $\frac{dv}{dr}=r$.

5. A right circular cylinder of volume v cu. ins. has height x inches and base radius r inches, prove that :—

(*i*) $\frac{dv}{dx}=\pi r^2$, if r is constant. (*ii*) $\frac{dv}{dr}=2\pi rx$, if x is constant.

6. The volnme v and the pressure p of a gas under constant temperature are connected by the relation

$$pv=c,$$

where c is a constant. Prove that

(*i*) $\frac{dp}{dv}=-\frac{c}{v^2}$. (*ii*) $\frac{dv}{dp}=-\frac{c}{p^2}$.

7. If $pv^{1.4}=c$ (a constant), find $\frac{dp}{dv}$.

8. Find $\frac{dy}{dx}$ when y is equal to

(*i*) $(3x+11)^2$. (*ii*) $(8x+5)^{\frac{1}{2}}$.

(*ii*) $\frac{1}{\sqrt{5x+3}}$. (*iv*) $(1-x)^{-1}$. (*v*) $\frac{1}{\sqrt[5]{(9-7x)^3}}$.

9. Differentiate *ab initio* $\sqrt{ax+o}$ (*P.U. 1952*)

CHAPTER III

GENERAL THEOREMS

Differentiation from first principles becomes very tedious in the case of complicated functions. With a view to simplify the process some general theorems are given in this chapter.

3·1. Theorem I. *An additive constant disappears in differenciation.*

Let $y=f(x)+c.$

When x becomes $x+\delta x$, let y become $y+\delta y$;

then
$$y+\delta y=f(x+\delta x)+c$$
$$\delta y=f(x+\delta x)-f(x)$$
$$\frac{\delta y}{\delta x}=\frac{f(x+\delta x)-f(x)}{\delta x}$$

In the limit when $\delta x \to 0$ we have

$$\frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\frac{f(x+\delta x)-f(x)}{\delta x}=\frac{d}{dx}[f(x)]$$

Substituting the value of y, it becomes

$$\frac{d}{dx}[f(x)+c)]=\frac{d}{dx}[f(x)]$$

i.e., the differential co-efficient of $f(x)+c$ is the same as that of $f(x)$. An additive constant may, therefore, be ignored in finding derivatives. This fact is expressed by saying that the *differential coefficient of a constant is zero.*

Examples

Ex. 1. The differential co-efficients of $5x^3$ and $5x^3+7$ are the same.

Ex. 2. $\frac{d}{dx}(a+x^2)=\frac{d}{dx}(x^2)=2x.$

3·2. Theorem II. *The differential co-efficient of the product of a constant and a function is the product of the constant and the differential co-efficient of the function.*

i.e., $$\frac{d}{dx}(cu)=c.\frac{du}{dx}$$

u, being a function of x.

When any increment δx is given to x, the corresponding increment in u is denoted by δu.

$\therefore$ if $y=cu$, $y+\delta y=c(u+\delta u)$

and $$\delta y=c(u+\delta u)-cu=c.\delta u,$$

$$\frac{\delta y}{\delta x}=c\,.\,\frac{\delta u}{\delta x}$$

$\therefore$ In the limit when $\delta x\rightarrow 0$

$$\frac{dy}{dx}=c\,.\,\frac{du}{dx}.$$

Hence $$\frac{d}{dx}(cu)=c.\frac{du}{dx}.$$

Examples

1. $\frac{d}{dx}(5x^3)=5\frac{d}{dx}(x^3)=5(3x^2)=15x^2.$

2. $\frac{d}{dr}(\pi r^2)=\pi\frac{d}{dr}(r^2)=\pi(2r)=2\pi r.$

3. If $s=a+\frac{1}{2}ft^2$, $\frac{ds}{dt}=0+\frac{1}{2}f.2t=ft.$

3.3. Theorem. *The differential co-efficient of the (algebraic) sum of any* **finite** *number of functions is the sum of their differential co-efficients.*

i.e., if u, v, w.........are functions of x,

then $$\frac{d}{dx}[u+v-w+\ldots\ldots\ldots]$$

$$=\frac{d}{dx}(u)+\frac{d}{dx}(v)-\frac{d}{dx}(w)+\ldots\ldots.$$

Let $y=u+v-w+\ldots\ldots\ldots$ (1)

Where $u, v, w, \ldots\ldots$ are functions of x, so that when x changes, u, v, w,...... etc., and also y shall change.

Let δu, δv, δw, ... and δy be the changes in u, v, w, ... and y corresponding to a change δx in x, then $y+\delta y=(u+\delta u)+(v+\delta v)-(w+\delta w)+$ ....(2)

Subtracting (1) from (2), we have

$$\delta y=\delta u+\delta v-\delta w+\ldots\ldots\ldots\ldots$$

$$\therefore\ \frac{\delta y}{\delta x}=\frac{\delta u}{\delta x}+\frac{\delta v}{\delta x}-\frac{\delta w}{\delta x}+\ldots\ldots$$

Now, when $\delta x\rightarrow 0$ and consequently δu, δv, δw,... and δy all tend to zero, we have

$$\underset{\delta x\rightarrow 0}{\text{Lt}}\ \frac{\delta y}{\delta x}=\underset{\delta x\rightarrow 0}{\text{Lt}}\ \frac{\delta u}{\delta x}+\underset{\delta x\rightarrow 0}{\text{Lt}}\ \frac{\delta v}{\delta x}-\underset{\delta x\rightarrow 0}{\text{Lt}}\ \frac{\delta w}{\delta x}+\ldots$$

$$\therefore\ \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}-\frac{dw}{dx}+\ldots\ldots$$

$$i.e.\ \frac{d}{dx}(u+v-w+\ldots\ldots)=\frac{d}{dx}(u)+\frac{d}{dx}(v)-\frac{d}{dx}(w)+\ldots$$

Note :—Here and in other theorems, u, v, w etc., are all assumed to be differentiable functions of x.

Examples

1. If $y=2x^3+3x^2-6x+7+12\sqrt{x}-3x^{-2}$

$$\text{then } \frac{dy}{dx}=\frac{d}{dx}(2x^3)+\frac{d}{dx}(3x^2)-\frac{d}{dx}(6x)$$

$$+\frac{d}{dx}(7)+\frac{d}{dx}(12x^{\frac{1}{2}})-\frac{d}{dx}(3x^{-2})$$

$$=2\cdot 3x^2+3\cdot 2x-6\cdot 1+0+12\left(\tfrac{1}{2}x^{-\frac{1}{2}}\right)-3(-2x^{-3})$$

$$=6\left(x^2+x-1+x^{-\frac{1}{2}}+x^{-3}\right)$$

2. Find the derivative of $\dfrac{x^3+3x+6}{\sqrt{x}}$

Let $y=(x^3+3x+6)/\sqrt{x}\ =x^{\frac{5}{2}}+3x^{\frac{1}{2}}+6x^{-\frac{1}{2}}$

✓ $$\therefore \quad \frac{dy}{dx} = \tfrac{5}{2}x^{\frac{3}{2}} + \tfrac{3}{2}x^{-\frac{1}{2}} + 6(-\tfrac{1}{2}x^{-\frac{3}{2}}) = \frac{5x^3+3x-6}{2x\sqrt{x}}$$

3. Differentiate $(x+c)^4$.

Let $y=(x+c)^4=x^4+4cx^3+6c^2x^2+4c^3x+c^4$

$$\therefore \quad \frac{dy}{dx} = 4x^3+4c\cdot 3x^2+6c^2\cdot 2x+4c^3\cdot 1+0$$

$$= 4(x^3+3cx^2+3c^2x+c^3)=4\,(x+c)^3.$$

Exercise V

Find the derivatives of the following with regard to the variables involved :—

1. (*i*) $\frac{1}{3}x^3+\frac{1}{2}x^2-x$ (*ii*) at^2+bt+c

(*iii*) $3+\dfrac{5}{y}-\dfrac{6}{y^2}$ (*iv*) $\dfrac{pz^2+qz+r}{\sqrt{z}}$.

2. (*i*) $(x+a)\,(x+b)$ (*ii*) $(y+3)^2\,(y+5)$

(*iii*) $8+2\sqrt{\theta}-\dfrac{2}{\sqrt{\theta}}$ (*iv*) $\dfrac{z+3}{z^{\frac{1}{3}}}$

3. Find $\dfrac{dy}{dx}$ if $y=x-\dfrac{x^3}{3\,!}+\dfrac{x^5}{5\,!}-\dfrac{x^7}{7\,!}$.

4. (*i*) If $f\,(\theta)=\theta^3-3\theta^2+3\theta-7$;
find $f'(3)$ and θ when $f'(\theta)=0$.

(*ii*) If $f\,(x)=x^4-10x^2-40\sqrt{x}$, find $f'(4)$.

5. (*i*) If $y=x^2-x$, prove that $\dfrac{dy}{dx}=\dfrac{2y}{x}+1$.

(*ii*) If $y=x^3-x^2$, prove that $\dfrac{dy}{dx}=x+\dfrac{3y}{x}$.

6. Prove that $y=x^{\frac{1}{2}}+x^{-\frac{1}{2}}$ satisfies the equation

$$2x\,\frac{dy}{dx}+y=2\sqrt{x} \qquad \text{(D. U.)}$$

7. A particle is moving along a straight line. The distance x ft. travelled in time t sec. is given by $x=\frac{1}{3}t^3-\frac{5}{2}t^2+4t$. Find the velocity $\dfrac{dx}{dt}$ at the end of two seconds and the time when the particle stops.

8. A body is projected vertically upwards. The height it has risen in time t is given by the formula $h=80t-16t^2$. Find the time which elapses before its upward velocity vanishes.

9. The distance s ft. of a certain body from the foot of an inclined plane at any time t sec. is given by $s=40+72t-8t^2$. Find when and where the body comes to rest.

3·31. In solved Ex. 3 Art 3·3, differentiation of $(x+c)^4$ was obtained by expanding it as the sum of a number of terms. Expansion of expressions such as $(x+c)^{151}$, $(x^2+3x+5)^{27}$ is neither easy nor convenient, and of expressions like $(3x+4)^{-\frac{3}{2}}$ is not possible in finite number of terms. A method of differentiating functions of this type is now given.

3·4. Function of a Function.

Sometimes we have an expression which is a function of a function of x. For example

$u=5x^2+2$ is a function of x.

$y=(5x^2+2)^3+3(5x^2+2)^2+8$ is a function of $5x^2+2$, so that y is a function of u, say $F(u)=u^3+3u+8$ where u is a function of x, say $f(x)=5x^2+2$.

3·5. Derivative of a Function of a Function.

Theorem IV If $y=F(u)$, a function of u and $u=f(x)$, a function of x, then

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

When x changes, u and y will also change.

Let δx be the increment of x and let δu, δy be the corresponding increments of u and y then

$$\frac{\delta y}{\delta x}=\frac{\delta y}{\delta u}\times\frac{\delta u}{\delta x}\text{[by simple multiplication]}$$

This is true because δx, δu, δy are finite quantities. Now when $\delta x\to 0$, then $\delta u\to 0$ and $\delta y\to 0$.

$$\therefore \underset{\delta x\to 0}{\text{Lt}}\frac{\delta y}{\delta x}=\underset{\delta u\to 0}{\text{Lt}}\frac{\delta y}{\delta u}\times\underset{\delta x\to 0}{\text{Lt}}\frac{\delta u}{\delta x}$$

$$\text{i.e. } \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Cor. If y is a function of u, u a function of v and v a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}$$

The formula can be easily extended.

Examples

Ex. 1. In the example mentioned above in Art. 3·4, we have $y=u^3+3u^2+8$ and $u=5x^2+2$;

we have $\frac{dy}{du}=3u^2+6u=3u(u+2)$ and $\frac{du}{dx}=10x$.

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 3u(u+2) \times 10x$$
$$= 30x(5x^2+2)(5x^2+4)$$

Ex. 2. If $y=(x^2+3x+5)^{27}$, find $\frac{dy}{dx}$

Put $u=x^2+3x+5$, then $y=u^{27}$

and $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 27u^{26} \times (2x+3)$
$$= 27(2x+3)(x^2+3x+5)^{26}$$

Ex. 3. Find $\frac{dy}{dx}$ when $y=u^3$, $u=\sqrt{v}$ and $v=5x^2+2$

Here $\frac{dy}{du}=3u^2=3v$, $\frac{du}{dv}=\frac{1}{2}v^{-\frac{1}{2}}=\frac{1}{2\sqrt{v}}$, and $\frac{dv}{dx}=10x$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx} = 3v \times \frac{1}{2\sqrt{v}} \times 10x = 15x\sqrt{v}$$
$$= 15x\sqrt{5x^2+2}$$

3·51. Putting $y=x$ in the result of Art 3·5, we have

$$\frac{dx}{dx} = \frac{dx}{du} \times \frac{du}{dx} \quad \text{i.e. } \frac{dx}{du} \times \frac{du}{dx} = 1 \text{ or } \frac{du}{dx} = 1 \Big/ \frac{dx}{du}$$

Thus $\frac{du}{dx}$ and $\frac{dx}{du}$ are reciprocals of each other and so are $\frac{dy}{dx}$ and $\frac{dx}{dy}$

This can be proved independently as follows :—

We have $\frac{\delta y}{\delta x} \cdot \frac{\delta x}{\delta y} = 1$

In the limit when $\delta x \to 0$ we have

$$\text{Lt} \frac{\delta y}{\delta x} \times \text{Lt} \frac{\delta x}{\delta y} = 1 \ \therefore\ \frac{dy}{dx} \times \frac{dx}{dy} = 1 \text{ or } \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

For example if $y = 3x + 4, \frac{dy}{dx} = 3$, and since $x = \frac{1}{3}(y-4), \frac{dx}{dy} = \frac{1}{3}$ Hence $\frac{dy}{dx} \times \frac{dx}{dy} = 3 \times \frac{1}{3} = 1$

3·52. Putting $y = u^n$ in the result of Art 3·5, we have $\frac{dy}{dx} = \frac{d}{du}(u^n) \times \frac{du}{dx} = nu^{n-1}\frac{du}{dx}$

It is important to note that the derivative of u^n with respect to u is nu^{n-1} but with respect to x, it is

$$nu^{n-1}\frac{du}{dx}$$

Thus $\frac{d}{dx}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dx}$ and $\frac{d}{dx}(\sqrt{u}) = \frac{1}{2\sqrt{u}} \cdot \frac{du}{dx}$

Examples

Ex. 1. Given $x^2 + y^2 = x + y$, find $\frac{dy}{dx}$

Differentiating both sides of the equation with respect to x, we have $2x + 2y\frac{dy}{dx} = 1 + \frac{dy}{dx}$

$$\therefore \quad 2y\frac{dy}{dx} - \frac{dy}{dx} = 1-2x \quad i.e. \ \frac{dy}{dx} = \frac{1-2x}{2y-1}$$

Note :—This is an example of finding the derivative of y with respect to x when y is an *implicit function* of x *i. e.* y is not explicitly expressed in terms of x.

Ex. 2. Differentiate x^7 with regard to x^3.

Let $y=x^7$ and $u=x^3$,

so that, $\frac{dy}{dx}=7x^6$ and $\frac{du}{dx}=3x^2$

then $\frac{dy}{du}=\frac{dy}{dx}\times\frac{dx}{du}=\frac{dy}{dx}\Big/\frac{du}{dx}=\frac{7x^6}{3x^2}=\frac{7}{3}x^4$

Ex. 3. If $y=t^3+3t+1$ and $x=t^2+5$, find $\frac{dy}{dx}$

$$\frac{dy}{dt}=3t^2+3=3\,(t^2+1) \text{ and } \frac{dx}{dt}=2t$$

$$\therefore \frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}=\frac{3(t^2+1)}{2t}=\tfrac{3}{2}(t+\frac{1}{t}).$$

Exercise VI

1. If $y=\frac{1}{u}$ and $u=5+4x^2$, find $\frac{dy}{dx}$ when $x=-\frac{1}{2}$

2. Differentiate the following with respect to the variables involved :—

(*i*) $(1+x+x^2)^3$ (*ii*) $\sqrt{t^2+1)}$ (*iii*) $(x+\frac{1}{x})^5$

(*iv*) $\sqrt{2ax-x^2}$ (*v*) $\frac{1}{2\sqrt{1-t^2}}$ (*vi*) $z+\sqrt{z^2+b^2}$

(*vii*) $(at^2+2bt+c)^{\frac{7}{2}}$ (*viii*) $\sqrt{1+\sqrt{1+x}}$

3. If $y=\sqrt{1+x^2}$, show that $y\frac{dy}{dx}=x$

4. Find $\frac{ds}{dt}$ when $s=2av+bv^2$ and $v=c+dt$

Q **5.** Verify that $\frac{dy}{dx} \times \frac{dx}{dy} = 1$

when (*i*) $y=ax+b$ (*ii*) $y=1/(x+5)$ (*iii*) $y=x^2+7$

6. Find $\frac{dy}{dx}$ when (*i*) $x=at^2,\ y=2at$

(*ii*) $x=1-t,\ y=t-t^2$ (*iii*) $x=5\theta^3+7,\ y=3\theta^2+\theta$.

7. If $v=\frac{4}{3}\pi r^3$ and $s=4\pi r^2$, show that $\frac{dv}{ds}=\frac{1}{2}r$.

8. Find $\frac{dy}{dx}$ for the following implicit functions of x:-

(*i*) $x^2+y^2=4x$ (*ii*) $x^3+y^3=x^2+y^2$ (*iii*) $y^2=4ax$
(*iv*) $x^2+y^2+2gx+2fy+c=0$ (*v*) $x^2/a^2+y^2/b^2=1$.

9. If $y=3z^2+\sqrt{z}+10z$ and $z=x^2+2x-7$, find $\frac{dy}{dx}$ when $x=2$

10. The velocity v of a body moving in a straight line is given by $v=\frac{27}{t^2-7t+15}$, find the acceleration $\frac{dv}{dt}$ when $t=3$.

11. If $y=\frac{1}{1+ax^2}+\frac{1}{1-ax^2}$, show that

$$\frac{dy}{dx}=\frac{8a^2x^3}{(1-a^2x^4)^2}.$$

12. Find $\frac{dy}{dx}$ when $y=z^3$, $z=\sqrt{1+u}$ and $u=\sqrt{x}$

13. Differentiate (*i*) $\sqrt{x}$ with respect to $x^{\frac{2}{3}}$

(*ii*) x^2 with respect to x^5

14. If $y=u^3-5u$ and $u=\sqrt{x^2-x}$, find $\frac{dy}{dx}$.

3·6. We are now in a position to simplify the method for obtaining the derivative of x^n when n is a negative integer or a fraction, positive or negative.

(*i*) Suppose n is a negative integer, say $-m$, and

$$y=x^n=x^{-m}=\frac{1}{x^m}.$$

then $x^m=y^{-1}$

Differentiating both sides with respect to x, we have

$$mx^{m-1}=-y^{-2}\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx}=-mx^{m-1}\times y^2=-mx^{m-1}\times x^{-2m}=-mx^{-m-1}$$

$$=nx^{n-1}$$

$$[\because \quad -m=n]$$

(*ii*) Suppose n is a fraction, say $\frac{p}{q}$ where p and q are integers

and $\qquad y=x^n=x^{\frac{p}{q}}$ ....(1)

then $\qquad y^q=x^p$

Differentiating both sides with respect to x, we have $\quad qy^{q-1}\frac{dy}{dx}=px^{p-1}$

$$\therefore \frac{dy}{dx}=\frac{p}{q}\,\frac{x^{p-1}}{y^{q-1}}=\frac{p}{q}x^{p-1}.\ y^{1-q}=\frac{p}{q}x^{p-1}.\ x^{\frac{p}{q}(1-q)}$$

[by (1)]

$$=\frac{p}{q}x^{p-1}\times x^{\frac{p}{q}-p}=\frac{p}{q}x^{\frac{p}{q}-1}=nx^{n-1}\left[\because \frac{p}{q}=n\right]$$

Ex. 1. If $y=(ax+b)^n$, find $\frac{dy}{dx}$ when n is any rational number.

Let $u=(ax+b)$, then $\frac{du}{dx}=a$ and $y=u^n$

$\therefore \dfrac{dy}{du}=nu^{n-1}$, where n may be any rational number.

Hence $\dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}=nu^{n-1}\times a=an(ax+b)^{n-1}$

[compare Art. 2·5]

Ex. 2. If $y=\dfrac{1}{\sqrt{4x+5}}=(4x+5)^{-\frac{1}{2}}$,

$$\frac{dy}{dx}=-\tfrac{1}{2}(4x+5)^{-\frac{3}{2}}\times 4=-2(4x+5)^{-\frac{3}{2}}$$

Ex. 3. Write down the derivatives of :—

$$(5x+6)^5,\ (1-x)^{23},\ \sqrt{ax+b},\ 1/(3x+4),\ (7+8x)^{\frac{2}{3}}$$

3·7. Derivative of a Product.

Theorem V. *The differential co-efficient of the product of two functions is the sum of the products of each function multiplied by the derivative of the other.*

i.e., $$\frac{d}{dx}(uv)=u\frac{d}{dx}(v)+v\frac{d}{dx}(u).$$

where u and v are functions of x.

Let $$y=uv.$$

When δx is the increment given to x, u, v and y will have corresponding increments ; let these be δu, δv and δy respectively.

Then
$$y+\delta y=(u+\delta u)(v+\delta v)$$
$$\delta y=(u+\delta u)(v+\delta v)-uv$$
$$=v\delta u+(u+\delta u)\delta v$$

$$\therefore \quad \frac{\delta y}{\delta x}=v\frac{\delta u}{\delta x}+(u+\delta u)\frac{\delta v}{\delta x}.$$

In the limit when $\delta x\to 0$, $u+\delta u\to u$ and we have

$$\frac{dy}{dx}=v\frac{du}{dx}+u\frac{dv}{dx},$$

i.e., $$\frac{d}{dx}(uv)=v\frac{du}{dx}+u\frac{dv}{dx}.$$

Cor. (*ii*) Putting $v=u$, we have $\frac{d}{dx}(u^2)=2u\frac{du}{dx}$.

(*i*) Putting $v=x$, we have $\frac{d}{dx}(ux)=x\frac{du}{dx}+u$.

3·71. We can readily extend the method to the product of three factors.

Suppose $y=uvw$.
Put $uv=z$.

Then $y=zw$, and $\frac{dy}{dx}=z\frac{dw}{dx}+w\frac{dz}{dx}$.

Substituting the value of z, we have

$$\frac{dy}{dx}=uv\frac{dw}{dx}+w\frac{d}{dx}(uv)$$

$$=uv\frac{dw}{dx}+w\left[u\frac{dv}{dx}+v\frac{du}{dx}\right]$$

$$=uv\frac{dw}{dx}+uw\frac{dv}{dx}+vw\frac{du}{dx}.$$

Dividing by $y=uvw$, it can also be written in the form.

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{w}\frac{dw}{dx}+\frac{1}{v}\frac{dv}{dx}+\frac{1}{u}\frac{du}{dx}.$$

The method can similarly be extended to the product of any *finite* number of functions.

Examples

Ex. 1. $y=(ax+b)(x^2+c)$

$$\frac{dy}{dx}=(x^2+c)\frac{d}{dx}(ax+b)+(ax+b)\frac{d}{dx}(x^2+c)$$

$$=(x^2+c)a+(ax+b).2x=ac+2bx+3ax^2$$

This can be verified by multiplying out and then differentiating.

Since $y=ax^3+bx^2+acx+bc$,

$\therefore$ $\frac{dy}{dx}=3ax^2+2bx+ac$.

Ex. 2. $y=(2x+3)(x^2-5)(7-x^2)$

$$\frac{dy}{dx}=(x^2-5)(7-x^2)\frac{d}{dx}(2x+3)+(2x+3)(7-x^2)$$

$$\times\frac{d}{dx}(x^2-5)+(2x+3)(x^2-5)\frac{d}{dx}(7-x^2)$$

$$=(x^2-5)(7-x^2).2+(2x+3)(7-x^2).2x$$
$$+(2x+3)(x^2-5)(-2x)$$
$$=2(x^2-5)(7-x^2)+2x(2x+3)(7-x^2)$$
$$-2x(2x+3)(x^2-5)$$
$$=2(x^2-5)(7-x^2)+2x(2x+3)(12-2x^2)$$
$$=-2(5x^4+6x^3-36x^2-36x\text{[illegible]}35)$$

Ex. 3. $y=(3x+8)^{\frac{7}{3}}(x+7)^{-3}$

$$\frac{dy}{dx}=(x+7)^{-3}\frac{d}{dx}(3x+8)^{\frac{7}{3}}+(3x+8)^{\frac{7}{3}}\frac{d}{dx}(x+7)^{-3}$$

$$=(x+7)^{-3}.\ \frac{7}{3}(3x+8)^{\frac{4}{3}}\cdot 3$$
$$+(3x+8)^{\frac{7}{3}}[-3(x+7)^{-4}]$$

$$=7(x+7)^{-3}(3x+8)^{\frac{4}{3}}-3(3x+8)^{\frac{7}{3}}(x+7)^{-4}$$

$$=(3x+8)^{\frac{4}{3}}(x+7)^{-4}[7(x+7)-3(3x+8)]$$

$$=(3x+8)^{\frac{4}{3}}(x+7)^{-4}(25-2x).$$

Ex. 4. If $x^2+y^2=3xy$, show that $\dfrac{dy}{dx}\times\dfrac{dx}{dy}=1$.

Differentiating with respect to x, we have

$$2x+2y\frac{dy}{dx}=3[1\cdot y+x\frac{dy}{dx}]\ i.e.,\frac{dy}{dx}=\frac{2x-3y}{3x-2y}$$

Now differentiating with respect to y, we get

$$2x\frac{dx}{dy}+2y=3[y\frac{dx}{dy}+x\cdot 1]\quad i.e.\ \frac{dx}{dy}=\frac{3x-2y}{2x-3y}$$

Hence $\dfrac{dy}{dx}\times\dfrac{dx}{dy}=\dfrac{2x-3y}{3x-2y}\times\dfrac{3x-2y}{2x-3y}=1.$

Exercise VII

1. Differentiate with respect to the variables involved :—

(*i*) $(2x+3)(7x+11)$ (*ii*) $(2x+1)(x^2+1)$

(*iii*) $(1+4x)^3(1-3x)$ (*iv*) $(y^2-1)(y^3+2y^2-3)$

(*v*) $(2+z^2)\sqrt{3-2z}$ (*vi*) $(1+t^2)\sqrt{1-t^2}$

2. (*i*) $x(x+2)(x^2-2)$ (*ii*) $\theta(\theta^2+3)(\theta^2-5)$

(*iii*) $(y+3)^2(y+5)(2y+3)$ (*iv*) $(2u^2+3u+1)(u^3+u+2)$

3. Find $\frac{dy}{dx}$ for the following implicit functions of x :—

(*i*) $x^3+y^3=3xy$ (*ii*) $x^5+y^5=5x^2y^2$

(*iii*) $ax^2+2hxy+by^2=1$ (*iv*) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$

4. If $y=z^4u^2$, find $\frac{dy}{dz}$ when $z=-2$, $u=1$, $\frac{du}{dz}=-1$.

5. Use the formula for differentiating a product to find $\frac{dy}{dx}$ when $y=x^m \cdot x^n$ and verify that it agrees with the result obtained by differentiating x^{m+n}.

6. Establish from first principles that

$$\text{if } ax^2+2hxy+by^2=1,\ \frac{dy}{dx}=-\frac{ax+hy}{hx+by}.$$

7. Prove that the product of the values of $\frac{dy}{dx}$ at the points common to the curves $y=\sqrt{a^2-x^2}$ and $xy=c^2$ is unity.

3·8. Derivative of a Quotient.

Theorem VI *If* $y=\frac{u}{v}$ *where u and v are functions of x, and v is not zero, then*

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

$$\text{Let } y=\frac{u}{v}=uv^{-1}$$

then by Art. 3·7, $\frac{dy}{dx}=v^{-1}\frac{du}{dx}+u\frac{d}{dx}\left(v^{-1}\right)$

$$=v^{-1}\frac{du}{dx}+u\left(-v^{-2}\frac{dv}{dx}\right)$$

$$=\frac{1}{v}\frac{du}{dx}-\frac{u}{v^2}\frac{dv}{dx}=\frac{1}{v^2}\left[v\frac{du}{dx}-u\frac{dv}{dx}\right]$$

3·81. The same result can be obtained from first principles as follows :—

Let $y=\frac{u}{v}$ then $y+\delta y=\frac{u+\delta u}{v+\delta v}$; $\delta y=\frac{u+\delta u}{v+\delta v}-\frac{u}{v}$

$$\therefore \frac{\delta y}{\delta x}=\frac{1}{\delta x}\left[\frac{u+\delta u}{v+\delta v}-\frac{u}{v}\right]$$

$$=\frac{1}{v(v+\delta v)}\left[\frac{uv+v\delta u-uv-u\delta v}{\delta x}\right]$$

$$=\frac{1}{v(v+\delta v)}\left[v\frac{\delta u}{\delta x}-u\frac{\delta v}{\delta x}\right]$$

In the limit as $\delta x\rightarrow 0$, $v+\delta v\rightarrow v$, and we get

$$\frac{dy}{dx}=\frac{1}{v^2}\left(v\frac{du}{dx}-u\frac{dv}{dx}\right)$$

$$\therefore \frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

i.e., the Derivative of the quotient of two functions

$$=\frac{\text{Den}^r.\ [\text{Derivative of Num}^r].-\text{Num}^r.\ [\text{Derivative of Den}^r.]}{\text{Square of Den}_r.}$$

Cor. (*i*) Putting $u=1$, we get, $\frac{d}{dx}\left(\frac{1}{v}\right)=-\frac{1}{v^2}\frac{dv}{dx}$.

(*ii*) Putting $u=1$ and $v=x$, we have $\frac{d}{dx}\left(\frac{1}{x}\right)=-\frac{1}{x^2}$

Examples

Ex. 1. $y=\dfrac{x^2-1}{x^2+1}$,

$$\frac{dy}{dx}=\frac{(x^2+1)\dfrac{d}{dx}(x^2-1)-(x^2-1)\dfrac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$=\frac{2x(x^2+1)-2x(x^2-1)}{(x^2+1)^2}=\frac{4x}{(x^2+1)^2}$$

Ex. 2. If $y=\sqrt{\dfrac{x^2-1}{x^2+1}}$ then $y^2=\dfrac{x^2-1}{x^2+1}$

Differentiating both sides with respect to x, we have

$$2y.\frac{dy}{dx}=\frac{d}{dx}\left[\frac{x^2-1}{x^2+1}\right]=\frac{4x}{(x^2+1)^2}\ \text{[by Ex. 1]}$$

$$\therefore\ \frac{dy}{dx}=\frac{2x}{y\,.\,(x^2+1)^2}=\frac{2x}{(x^2+1)^2}\sqrt{\frac{x^2+1}{x^2-1}}=\frac{2x}{(x^2+1)\sqrt{x^4-1}}$$

Ex. 3. $y=(\sqrt{x}-\sqrt{b})/(\sqrt{x}+\sqrt{b})$

$$\frac{dy}{dx}=\frac{(\sqrt{x}+\sqrt{b})\dfrac{d}{dx}(\sqrt{x}-\sqrt{b})-(\sqrt{x}-\sqrt{b})\dfrac{d}{dx}(\sqrt{x}+\sqrt{b})}{(\sqrt{x}+\sqrt{b})^2}$$

$$=\frac{1}{(\sqrt{x}+\sqrt{b})^2}\left[(\sqrt{x}+\sqrt{b})\,\frac{1}{2\sqrt{x}}-(\sqrt{x}-\sqrt{b})\,\frac{1}{2\sqrt{x}}\right]$$

$$=\frac{\sqrt{b}}{\sqrt{x}(\sqrt{x}+\sqrt{b})^2}$$

3·9 Formulæ for Differentiation. The formulae derived in this chapter are collected in the following table for reference :

1. $\dfrac{dc}{dx}=0$

2. $\dfrac{d}{dx}(x^n)=nx^{n-1}$, $\qquad \dfrac{d}{dx}(u^n)=nu^{n-1}\dfrac{du}{dx}$

3. $\dfrac{d}{dx}[u+v-w+\ldots\ldots]=\dfrac{du}{dx}+\dfrac{dv}{dx}-\dfrac{dw}{dx}+\ldots$

4. $$\frac{d}{dx}(uv)=v\frac{du}{dx}+u\frac{dv}{dx}.$$

5. $$\frac{d}{dx}\left[\frac{u}{v}\right]=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}.$$

6. $$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}.$$

Exercise VIII

Differentiate :—

1. (i) $\frac{ax+b}{cx+d}$ (ii) $\frac{x}{x^2+1}$ (iii) $\frac{x^n}{1+x^n}$ (iv) $-\frac{1-x+x^2}{1+x+x^2}$

(v) $\frac{(x+1)^3}{(x-1)^2}$ (vi) $\frac{1+2x}{(1-x)^3}$ (vii) $\frac{1+4x}{4(1+2x)^2}$ (viii) $\frac{x^3-x}{1+x-3x^2}$

2. (i) $\frac{1-\sqrt{x}}{1+\sqrt{x}}$ (ii) $\frac{\sqrt{a^2-x^4}}{x}$ (iii) $\frac{x}{\sqrt{a^2-x^2}}$

(iv) $\frac{a}{x+\sqrt{a^2+x^2}}$ (v) $\sqrt{\frac{1-x}{1+x}}$ (vi) $\sqrt{\frac{x}{1-2x}}$

(vii) $\left(\frac{1-x}{1+x}\right)^3$ (viii) $\frac{1}{(x^2-3)(x-3)}$

3. Differentiate with respect to x

$(\sqrt{a+x}-\sqrt{a-x})/(\sqrt{a+x}+\sqrt{a-x})$ (*D.U.*)

4. (i) If $y=\frac{x}{x-8}$, prove that $x\frac{dy}{dx}=y(1-y)$

(ii) If $y=\frac{x}{\sqrt{1+x}}$, prove that $x^3\frac{dy}{dx}=(1+\frac{1}{2}x)y^3$

(iii) If $y=x\pm\sqrt{x^2+1}$, show that $(y-x)\frac{d}{dx}=y.$

5. Express $\frac{dy}{dx}$ in terms of t, when

(i) $x=\frac{2t}{1+t^2}$, $y=\frac{1-t^2}{1+t^2}$ (D. U.)

(ii) $x=\dfrac{a\,(1-t^2)}{1+t^2},\ y=\dfrac{2bt}{1+t^2}$ (P. U. 1941)

6. Prove the formula $\dfrac{dx}{dy}=1\Big/\dfrac{dy}{dx}$ and apply it to find $\dfrac{dx}{dy}$ when x and y are connected by the relation

$$y=\sqrt{\frac{1+x}{2+x}}$$ (D. U.)

7. Differentiate (i) $\dfrac{x^2}{1+x^2}$ with respect to x^2

(P. U 1944)

(ii) $\dfrac{ax+b}{cx+d}$ with respect to $\dfrac{a'x+b'}{c'x+d'}$ (D. U.)

8. Assuming that $\dfrac{1-x^{n+1}}{1-x}=1+x+x^2+\dots\dots+x^n$

deduce, by differentiation, the sum of the series
$1+2x+3x^2+\dots\dots\dots+nx^{n-1}$

9. Differentiate the following :—

(i) $\dfrac{x}{a-\sqrt{a^2-x^2}}$ (P. U.) (ii) $\dfrac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}$

(P. U. 1941)

(iii) $\dfrac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}$ (P.U.) (iv) $\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$

[**Hint** : Rationalise the denominator in each case]

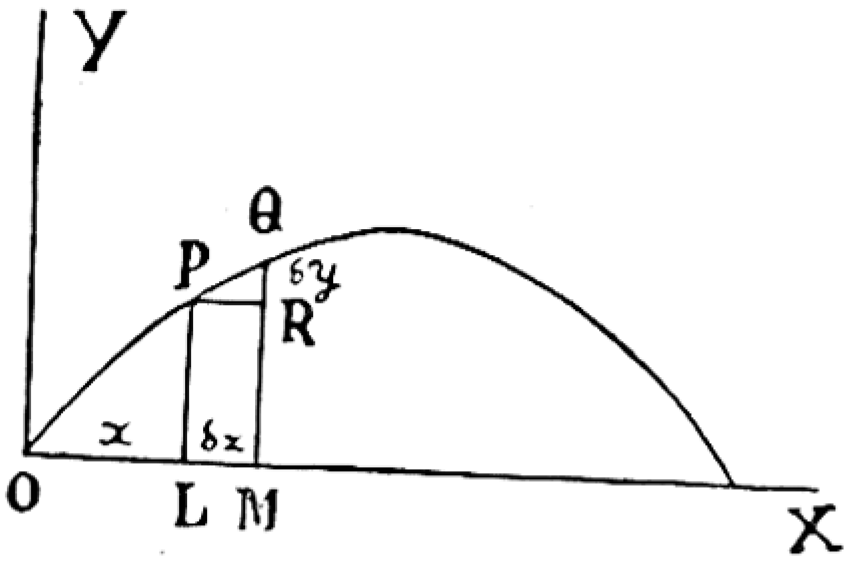
10. Differentiate $\sqrt{\dfrac{x^2-2ax}{bx-cx^2}}$ (P. U. 1952)

CHAPTER IV

DIFFERENTIATION OF CIRCULAR FUNCTIONS

Note.—The angles are all taken in circular measure unless otherwise stated.

4.1 Differentiation of sin x, where x is given in radians.



Let $y = LP = \sin x$,
then $y + \delta y = MQ$
$= \sin(x + \delta x)$
$\delta y = RQ$
$= \sin(x + \delta x) - \sin x$

$$\frac{\delta y}{\delta x} = \frac{RQ}{PR}$$

$$= \frac{\sin(x + \delta x) - \sin x}{\delta x}$$

$$= \frac{2\cos\left(x + \frac{\delta x}{2}\right)\sin\frac{\delta x}{2}}{\delta x}$$

$$= \cos\left(x + \frac{\delta x}{2}\right) \cdot \frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}.$$

In the limit when $\delta x \to 0$, we have

$$\frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \cos\left(x + \frac{\delta x}{2}\right) \underset{\delta x \to 0}{\text{Lt}} \left\{ \frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}} \right\}$$

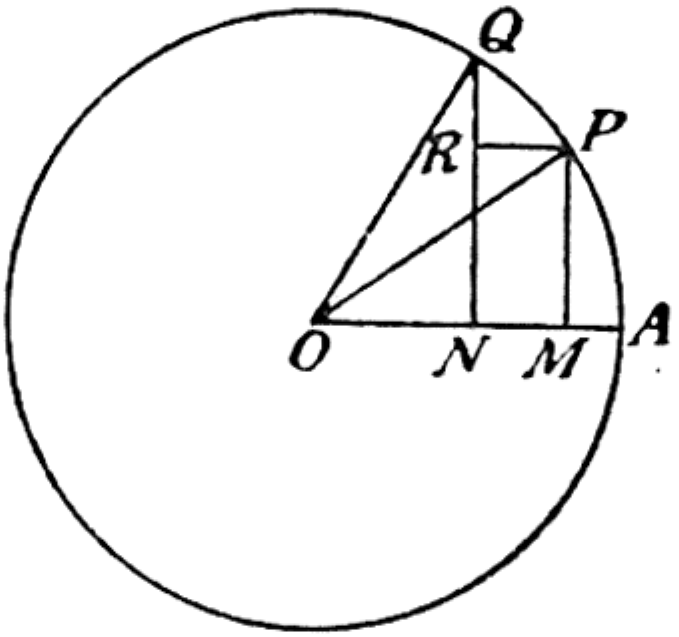
$$= \cos x . 1,$$

The limit of the second term on the right hand side is 1, for, the angle x and consequently δx is in circular measure and by Trigonometry $\underset{\theta \to 0}{\text{Lt}} \frac{\sin \theta}{\theta} = 1$ when θ is in radian measure.

Hence $$\frac{d}{dx}(\sin x) = \cos x$$

If u is any function of x, $\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$

4·1 *Geometrical. Let $OA = OP = OQ = 1$.



$\angle AOP = x$ (in circular measure)

$$= \frac{\text{arc } AP}{OA} = \text{arc } AP$$

$\angle AOQ = x + \delta x = \text{arc } PQ$, then $\delta x = \angle POQ = \text{arc } PQ$.

and $\delta y = \sin(x + \delta x) - \sin x$

$$= \frac{NQ}{OQ} - \frac{MP}{OP}$$

$$= NQ - MP = RQ.$$

$$\therefore \frac{\delta y}{\delta x} = \frac{RQ}{\text{arc } PQ}$$

$$= \frac{RQ}{\text{chord } PQ} \times \frac{\text{chord } PQ}{\text{arc } PQ}$$

$$\therefore \frac{dy}{dx} = \text{Lt} \left[\left(\frac{RQ}{\text{chord } PQ} \right) \left(\frac{\text{chord } PQ}{\text{arc } PQ} \right) \right]$$

$$= \cos \angle PQR .\ 1 = \cos \angle RPO = \cos \angle AOP$$

$$= \cos x$$

For, in the limit when $\delta x = \angle POQ \to 0$, $\angle OPQ \to 90^\circ$, and $\angle PQR \to \angle RPO = \angle AOP = x$.

4·2. Differentiation of cos x

Let $y = \cos x$.

$y + \delta y = \cos(x + \delta x)$

$$\therefore \frac{\delta y}{\delta x} = \frac{\cos(x + \delta x) - \cos x}{\delta x}$$

$$= \frac{-2 \sin \frac{\delta x}{2} \sin\left(x + \frac{\delta x}{2}\right)}{\delta x}$$

$$\therefore \quad \frac{dy}{dx} = -\underset{\delta x \to 0}{\mathrm{Lt}} \left\{ \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}} \right\} \underset{\delta x \to 0}{\mathrm{Lt}} \sin\left(x + \frac{\delta x}{2} \right)$$

$$= -1 . \sin x = -\sin x.$$

i.e. $\mathbf{\frac{d}{dx}(\cos x) = -\sin x}$

If u is any function of x,

$$\frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx}$$

Note ;—The derivative of cos x can be deduced from that of sin x as follows :—

$$\frac{d}{dx}(\cos) = \frac{d}{dx} \sin\left(\frac{\pi}{2} - x \right) = \cos\left(\frac{\pi}{2} - x \right) \times (-1)$$

$$= -\sin x$$

*Ex. Find the differential co-efficient of cos x geometrically.

4.3. Differentiation of tan x.

Let $y = \tan x$

$$y + \delta y = \tan (x + \delta x)$$

$$\delta y = \tan (x + \delta x) - \tan x = \frac{\sin (x + \delta x)}{\cos (x + \delta x)} - \frac{\sin x}{\cos x}$$

$$= \frac{\sin (x + \delta x) \cos x - \cos (x + \delta x) \sin x}{\cos (x + \delta x) \cos x}$$

$$= \frac{\sin (x + \delta x - x)}{\cos (x + \delta x) \cos x}$$

$$\frac{\delta y}{\delta x} = \frac{\sin \delta x}{\delta x} . \frac{1}{\cos (x + \delta x) \cos x}$$

$$\therefore \frac{dy}{dx} = \underset{\delta x \to 0}{\mathrm{Lt}} \left(\frac{\sin \delta x}{\delta x} \right) . \ \mathrm{Lt} \frac{1}{\cos (x + \delta x) \cos x}$$

$$= 1 . \frac{1}{\cos^2 x} = \sec^2 x.$$

Hence $\mathbf{\frac{d}{dx}(\tan x) = \sec^2 x.}$

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If u is any function of x,

$$\frac{\mathbf{d}}{\mathbf{dx}}\ (\tan u) = \sec^2 u\ \frac{\mathbf{du}}{\mathbf{dx}}.$$

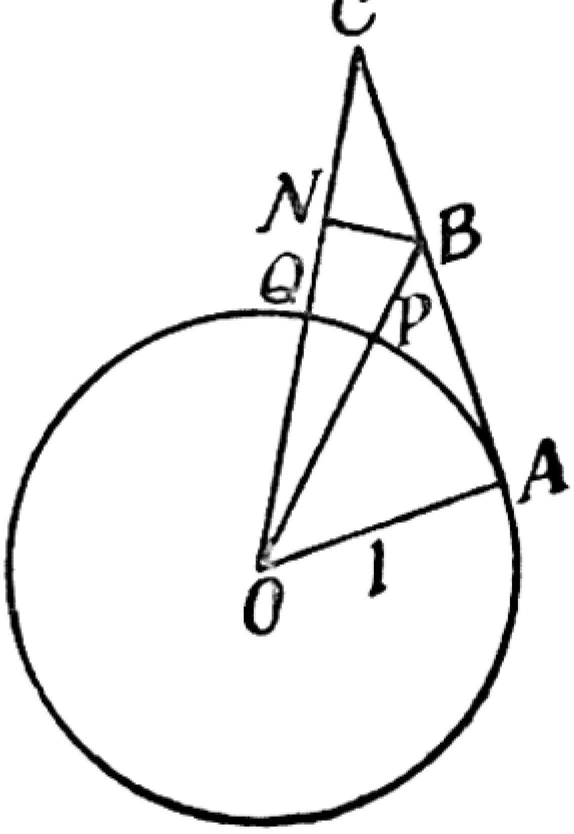
4.31. *Geometrical – Let $OA = OP = OQ = 1$.

$\angle AOP = x$ radians $=$ ArcAP,

$\angle AOQ = x + \delta x =$ arc AQ,

then $\angle POQ =$ increment of x

$= \delta x =$ arc PQ.



OP, OQ meet the tangent at A in B and C and $BN \perp OC$. In the limit when Q coincides with P, C coincides with B and $\angle OCA$ becomes equal to $\angle OBA = \frac{\pi}{2} - x$.

Now $y = \tan x$

$= AB/OA = AB$

and $y + \delta y = \tan (x + \delta x)$.

$= AC/OA = AC$

$\therefore \quad \delta y = AC - AB = BC$

$$\frac{\delta y}{\delta x} = \frac{BC}{PQ} = \frac{BC.BN}{BN.PQ}$$

$$\therefore \quad \frac{dy}{dx} = \text{Lt}\ \frac{BC}{BN} \times \text{Lt}\ \frac{BN}{PQ}$$

$$= \text{Lt}\ \operatorname{cosec} \angle BCN \times \text{Lt}\ \frac{OB}{OP}$$

$$= \text{Lt}\ \operatorname{cosec}\left(\frac{\pi}{2} - x\right) \times \text{Lt}\ \frac{OB}{OA}$$

$$= \sec x \times \sec x = \sec^2 x.$$

4·32. *Derivative of tan x deduced from those of sin x and cos x.*

Let $y = \tan x = \sin x / \cos x$.

$$\text{then}\quad \frac{dy}{dx} = \frac{\cos x\ \frac{d}{dx}(\sin x) - \sin x\ \frac{d}{dx}(\cos x).}{\cos^2 x}$$

$$= \frac{\cos x\ (\cos x) - \sin x\ (-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

4·4. Differentiation of cot x.

Let $y=\cot x$

then $y+\delta y=\cot(x+\delta x)$

and $$\delta y=\cot(x+\delta x)-\cot x=\frac{\cos(x+\delta x)}{\sin(x+\delta x)}-\frac{\cos x}{\sin x}$$

$$=\frac{\sin x\cos(x+\delta x)-\sin(x+\delta x)\cos x}{\sin(x+\delta x)\sin x}$$

$$=\frac{\sin(x-x-\delta x)}{\sin(x+\delta x)\sin x}$$

$$\therefore\quad\frac{\delta y}{\delta x}=-\frac{\sin\delta x}{\delta x}\cdot\frac{1}{\sin(x+\delta x)\sin x}$$

and $$\frac{dy}{dx}=\underset{\delta x\to 0}{\mathrm{Lt}}\frac{\delta y}{\delta x}$$

$$=-\underset{\delta x\to 0}{\mathrm{Lt}}\left(\frac{\sin\delta x}{\delta x}\right)\underset{\delta x\to 0}{\mathrm{Lt}}\frac{1}{\sin(x+\delta x)\sin x}=-1.\operatorname{cosec}^2 x$$

$$=-\operatorname{cosec}^2 x.$$

Hence $$\mathbf{\frac{d}{dx}(\cot x)=-\operatorname{cosec}^2 x}$$

and $$\mathbf{\frac{d}{dx}(\cot u)=-\operatorname{Cosec}^2 u\,\frac{du}{dx}}$$

Ex. Differentiate $\cot x$ regarding it as $\cos x/\sin x$.

4.5. Differentiation of sec x.

Let $y=\sec x$.

$y+\delta y=\sec(x+\delta x)$

$\delta y=\sec(x+\delta x)-\sec x$

$$=\frac{\cos x-\cos(x+\delta x)}{\cos(x+\delta x)\cos x}=\frac{2\sin\frac{\delta x}{2}\sin\left(x+\frac{\delta x}{2}\right)}{\cos x\cos(x+\delta x)}$$

$$\therefore\ \frac{\delta y}{\delta x}=\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}\times\frac{\sin\left(x+\frac{\delta x}{2}\right)}{\cos x\cos(x+\delta x)}$$

$$\therefore \frac{dy}{dx}=\underset{\delta x\to 0}{\mathrm{Lt}}\left\{\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}\right\}\underset{\delta x\to 0}{\mathrm{Lt}}\ \frac{\sin\left(x+\frac{\delta x}{2}\right)}{\cos x\cos(x+\delta x)}$$

$$=1.\frac{\sin x}{\cos x\cos x}=\tan x\sec x$$

i.e., $\frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{sec\ x})=\mathbf{sec\ x\ tan\ x}.$

and $\frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{sec\ u})=\mathbf{sec\ u\ tan\ u}\frac{\mathbf{du}}{\mathbf{dx}}$

Proceeding as above we can obtain

$$\frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{cosec\ x})=-\mathbf{cosec\ x\ cot\ x}.$$

Ex. Differentiate sec x and cosec x from the fact that sec $x=(\cos x)^{-1}$ and cosec $x=(\sin x)^{-1}$.

Examples

Ex 1. Differentiate $\sin(9x+5)$.

Let $y=\sin(9x+5)$ and $u=9x+5$
then $y=\sin u$

$$\therefore \frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{d}{du}(\sin u)\frac{d}{dx}(9x+5)=\cos u\times 9$$

$$=9\cos(9x+5).$$

Ex 2. Differentiate $\sin x^\circ$.

Let $y=\sin x^\circ=\sin\left[\frac{\pi x}{180}\text{ radians}\right]$

$$\therefore \frac{dy}{dx}=\cos\frac{\pi x}{180}\times\frac{d}{dx}\left(\frac{\pi x}{180}\right)$$

$$=\frac{\pi}{180}\cos\frac{\pi x}{180}=\frac{\pi}{180}\cos x^\circ.$$

Ex. 3. Differentiate $\sin 3x\sin 7x$

Let $y=\sin 3x\sin 7x=\frac{1}{2}(\cos 4x-\cos 10x)$

$$\therefore \frac{dy}{dx}=\tfrac{1}{2}[-4\sin 4x-(-10\sin 10x)]$$

$$=5\sin 10x-2\sin 4x.$$

Ex. 4. Find the derivative of $\cos x^2$.

Let $y=\cos x^2$ and let $u=x^2$, then $y=\cos u$

$$\text{then } \frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=-\sin u\times 2x=-2x\sin x^2.$$

Ex. 5. Find $\dfrac{dy}{dx}$ if $y=x^2\cos x^2$.

$$\frac{dy}{dx}=\cos x^2\frac{d}{dx}(x^2)+x^2\frac{d}{dx}(\cos x^2)$$

$$= 2x\cos x^2+x^2(-2x\sin x^2) \qquad \text{[By Ex. 4]}$$

$$= 2x\cos x^2-2x^3\sin x^2.$$

Ex. 6. If $y=\tan^2 x\sec 2x$, find $\dfrac{dy}{dx}$.

$$\frac{dy}{dx}=\sec 2x\frac{d}{dx}(\tan^2 x)+\tan^2 x\frac{d}{dx}(\sec 2x)$$

$$=\sec 2x\,[2\tan x.\sec^2 x]+\tan^2 x\,[2\sec 2x\tan 2x].$$

$$=2\tan x\sec 2x\,(\sec^2 x+\tan x\tan 2x).$$

Ex. 7. Find $\dfrac{dy}{dx}$ when $y=\dfrac{\sin x-\cos x}{\sin x+\cos x}$.

$$\frac{dy}{dx}=\frac{(\sin x+\cos x)\frac{d}{dx}(\sin x-\cos x)-(\sin x-\cos x)\frac{d}{dx}(\sin x+\cos x)}{(\sin x+\cos x)^2}$$

$$=\frac{(\sin x+\cos x)\,[\cos x+\sin x]-(\sin x-\cos x)\,[\cos x-\sin x]}{(\sin x+\cos x)^2}$$

$$=\frac{(\sin x+\cos)^2+(\sin x-\cos x)^2}{\sin^2 x+\cos^2 x+2\sin x\cos x}$$

$$=\frac{2(\sin^2 x+\cos^2 x)}{1+2\sin x\cos x}=\frac{2}{1+\sin 2x}.$$

Ex. 8. Find the differential coefficient of $\sec(\sin x)$.
Let $u=\sin x$ and $y=\sec(\sin x)=\sec u$.

$$\text{then } \frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\sec u\tan u\times\frac{d}{dx}(\sin x)$$

$$=\cos x\sec u\tan u=\cos x\sec(\sin x)\tan(\sin x).$$

Exercise IX

1. Find the derivatives of the following with respect to the independent variable :

(*i*) $\text{Cos } 3x$ (*ii*) $\text{Sin } (c-nz)$ (*iii*) $\text{Cot } (180^\circ-7x^\circ)$
(*iv*) $\text{Sin}^2 3x$ (*v*) $\text{Cos}^2 \frac{1}{2}x$ (*vi*) $x^n \text{ Sin } nx$
(*vii*) $2 \tan\sqrt{y}$ (*viii*) $\text{Cosec}^4 \frac{1}{2}t$
(*ix*) $\sqrt{\text{Sin } \theta}$ (*x*) $\tan^{\frac{3}{2}} z-3\tan^{\frac{1}{2}} z.$

A

2. (*i*) $\text{Sin } 3z \text{ Cos } 2z$ (*ii*) $\text{Sin } x \text{ Cos}^3 x$ (*iii*) $\tan x \text{ Sin } x$ (*iv*) $\text{Sin } 3x\sqrt{\text{Cos } 3x}$ (*v*) $\sqrt{\theta} \text{ Cos } \frac{1}{2}\theta$ (*iv*) $\tan \sqrt{x}-\text{Cot}\sqrt{x}$ (*vii*) $\sin\theta \cos\theta \cos 2\theta$ (*viii*) vers x (*vi*) covers x.

3. (*i*) $\dfrac{1+\text{Sin } x}{\text{Cos } x}.$ (*ii*) $\dfrac{1-\text{Cos}\theta}{1+\text{Cos}\theta}.$ (*iii*) $\dfrac{1+\text{Sin } \frac{1}{2}z}{1-\text{Sin } \frac{1}{2}z}$

(*iv*) $\dfrac{\text{Sec } x+\tan x}{\text{Sec } x-\tan x}.$ (*v*) $\dfrac{x+\text{Sin } x}{1-\text{Cos } x}.$ (*vi*) $\dfrac{\tan z}{1-\tan^2 z}$

(*vii*) $\dfrac{\sin 2x}{\sqrt{x}}.$ (*viii*) $\dfrac{\sin 2z}{4+z^2}.$

4. (*a*) (*i*) $\text{Cos } (m \cot x)$ (*ii*) $\text{Sin } (\text{Sin } x-\text{Cos } x)$

(*b*) $\text{Cos } (\text{Sin}\sqrt{ax+b})$ (P. U. 1953)

5. (*i*) $\sqrt{\text{Sin } x}$ (*ii*) $\text{Sin } \sqrt{z}$ (*iii*) $\sqrt{\text{Sin}\sqrt{x}}$
(P. U. 1952)

A

6. (*a*) Show that $\dfrac{d}{dx}$ $(\text{Sec}^2 nx-\tan^2 nx)=0$

and account for this result.

(*b*) Differentiate $\text{Sec}^4 nx-\tan^4 nx.$

7. Find $\dfrac{dy}{dx}$ if (*i*) $x=a \text{ Cos } \phi$, $y=b \text{ Sin } \phi.$

(*ii*) $x=a(\theta+\text{Sin}\theta)$, $y=a(1-\text{Cos } \theta)$
(P. U. 1953)

(*iii*) $x=a \text{ Cos}^3 t,$ $y=b \text{ Sin}^3 t.$

8. Find $\dfrac{dy}{dx}$ if (*i*) $x=\tan (x+y)$, (*ii*) $xy=\sin^2 x$

(*iii*) $x \sin y-y \sin x=0$

A (*iv*) If $\tan\theta=\dfrac{y}{x}$, show that $\dfrac{d\theta}{dt}=\left(x\dfrac{dy}{dt}-y \dfrac{dx}{dt}\right) \Big/ (x^2+y^2),$ x and y being functions of t.

A

9. Differentiate the trigonometric formulae for
(*i*) $\sin 2x$ (*ii*) $\sin 3x$ (*iii*) $\cos(x+a)$ (*iv*) $\tan 2x$.
and obtain the corresponding formulae for
(*i*) $\cos 2x$ (*ii*) $\cos 3x$ (*iii*) $\sin(x+a)$ (*iv*) $\sec 2x$.

10. Differentiate (*i*) $\cos x$ with respect to $\cot x$
(*ii*) $\sin x^2$ with regard to x^2.
(*iii*) $\dfrac{\sin x}{x}$ with respect to $\dfrac{x}{\sin x}$.

11. Show that the derivatives of all trigonometrical ratios are deducible from the derivative of $\sin x$.

***4·6. Differentiation from first principles.** It means that the derivative of a function is to be obtained directly from the definition of differential coefficient as has been already done in the case of simple functions, without the application of the general theorems on differentiation or the results of standard functions already derived. A few examples of trigonometric functions are given below :—

Ex. 1. Differentiate from first principles $\sin^2 x$.

Let $y=\sin^2 x=\frac{1}{2}(1-\cos 2x)=\frac{1}{2}-\frac{1}{2}\cos 2x$,
then $y+\delta y=\frac{1}{2}-\frac{1}{2}\cos 2(x+\delta x)$
and $\delta y=\frac{1}{2}[\cos 2x-\cos 2(x+\delta x)]$
$=\frac{1}{2}\times 2\sin(2x+\delta x)\sin\delta x$

$$\therefore \frac{dy}{dx}=\underset{\delta x\to 0}{Lt}\frac{\delta y}{\delta x}=Lt\,\sin(2x+\delta x)\times Lt\,\frac{\sin\delta x}{\delta x}=\sin 2x.$$

Ex. 2. Find *ab initio* the derivative of $\sin x^2$

Let $y=\sin x^2$, then $y+\delta y=\sin(x+\delta x)^2$

$\therefore \delta y=\sin(x+\delta x)^2-\sin x^2$
$=2\cos\frac{1}{2}[(x+\delta x)^2+x^2]\sin\frac{1}{2}[(x+\delta x)^2-x^2]$
$=2\cos[x^2+x\delta x+\frac{1}{2}(\delta x)^2]\sin[x\delta x+\frac{1}{2}(\delta x)^2]$

$$\therefore \frac{\delta y}{\delta x}=2\cos[x^2+x\delta x+\tfrac{1}{2}(\delta x)^2]\,\frac{\sin[x\delta x+\frac{1}{2}(\delta x)^2]}{x\delta x+\frac{1}{2}(\delta x)^2}\times\frac{x\delta x+\frac{1}{2}(\delta x)^2}{\delta x}.$$

$$\text{Hence } \frac{dy}{dx}=\underset{\delta x\to 0}{Lt}\frac{\delta y}{\delta x}=2\cos x^2\times 1\times x=2x\cos x^2.$$

Ex. [illegible] Find from first principles, the differential coefficient of $\sqrt{\cos x}$.

Let $y=\sqrt{\cos x}$, then $y+\delta y=\sqrt{\cos(x+\delta x)}$

$$\therefore \delta y=\sqrt{\cos(x+\delta x)}-\sqrt{\cos x}=\frac{\cos(x+\delta x)-\cos x}{\sqrt{\cos(x+\delta x)}+\sqrt{\cos x}}$$

$$=\frac{-2\sin(x+\frac{1}{2}\delta x)\sin\frac{1}{2}\delta x}{\sqrt{\cos(x+\delta x)}+\sqrt{\cos x}}$$

$$\therefore \frac{dy}{dx}=\underset{\delta x\to 0}{Lt}\frac{\delta y}{\delta x}$$

$$=-\underset{\delta x\to 0}{Lt}\frac{\sin(x+\frac{1}{2}x\delta)}{\sqrt{\cos(x+\delta x)}+\sqrt{\cos x}}\times\underset{\delta x\to 0}{Lt}\frac{\sin\frac{1}{2}\delta x}{\frac{1}{2}\delta x}$$

$$=-\frac{\sin x}{2\sqrt{\cos x}}.$$

*Exercise IX A

1. Differentiate from first principles:—
   (i) $\sin 2\theta$. (ii) $\cos \frac{1}{2} x$. (iii) $\cos^2 x$.
   (iv) $\sin^3 x$. (v) $\sin x \cos x$. (vi) $\cos x^2$.
   (vii) $\sin \sqrt{x}$.

2. Differentiate from first principles $\tan x^2$. (D. U. 1938)

3. Find from definition the differential coefficients of (i) x^n (ii) $\sqrt{\sin x}$. (P. U. 1953)

4. Differentiate with respect to x :—
   (i) $\sin \bar{x}$ (ii) $\operatorname{cosec}(a\theta+b)$ (iii) $\cos\sqrt{y}$.

4·7. Inverse trigonometric functions. In the equation

$$x=\sin y$$

x is the sine of y, or, what is exactly the same thing, y is *an angle whose sine is* x.

When solved for y, the equation is written as

$$\mathbf{y=\sin^{-1}x} \quad \text{or} \quad \mathbf{y=arc\ sin\ x}.$$

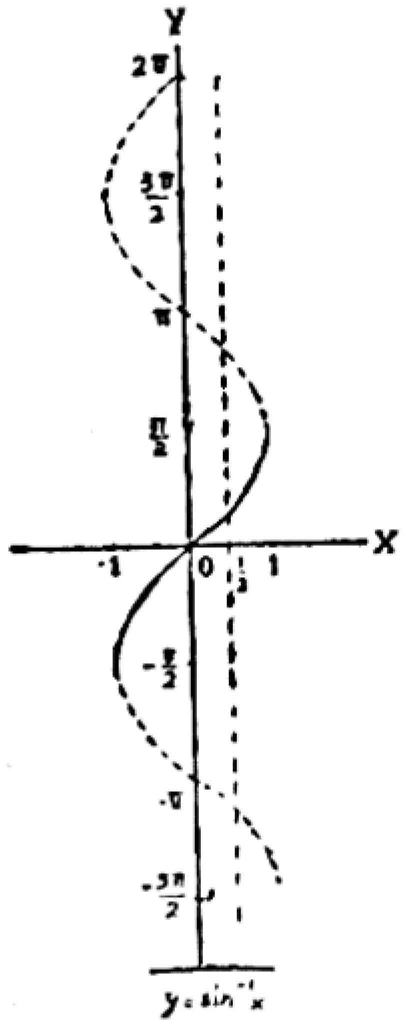
Similarly $\mathbf{y=\cos^{-1}x}$ if $\mathbf{\cos y=x}$ and $\mathbf{y=\tan^{-1}x}$ if $\mathbf{\tan y=x}$, etc.

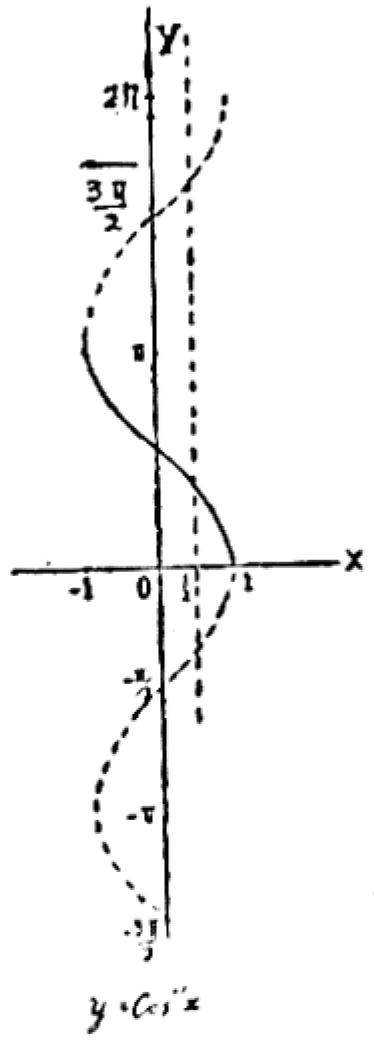
The functions $\sin^{-1}x$ etc. are called *inverse trigonometric functions*.

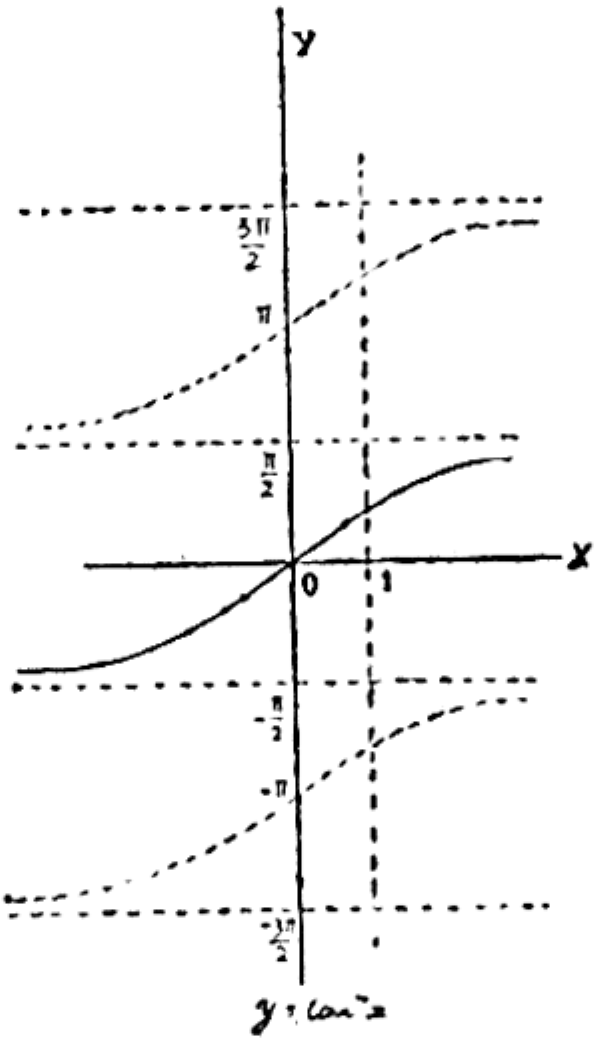
$\sin^{-1}x$ does not mean $\sin x$ to the power -1, which is written as $(\sin x)^{-1}$.

For any particular value of x, y has an infinite number of values, for example when $x=\frac{1}{2}$, the angle y whose sine is $\frac{1}{2}$ *i.e.* $\sin^{-1}\frac{1}{2}$ is not only $\pi/6$ or 30° but may be also $5\pi/6$ or 150°, $2\pi+\pi/6$, $3\pi-\pi/6$ etc., in fact an infinite number of values.

Thus inverse trigonometric functions are infinitely many valued *i.e.*, corresponding to a given value of the independent variable x, there are an infinite values of the function y. This is made clear by the graphs of $\sin^{-1}x$, $\cos^{-1}x$ and $\tan^{-1}x$ shown below.







$y=\sin^{-1}x$ $y=\cos^{-1}x$ $y=\tan^{-1}x$

In the graphs of $\sin^{-1}x$, $\cos^{-1}x$, the dotted line $x=\frac{1}{2}$ meets the curves in an infinite number of points, so does the line $x=1$ in the graph of $\tan^{-1}x$.

In order to avoid discrepancies of sign in the formulae of differentiation of such functions, the functions are made single valued by restricting them to a single branch of each fuction.

In the graph of $\sin^{-1}x$, for any value of x from -1 to 1, there is one and only one value of y, if y is confined to the heavily marked branch between $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$ and this value, called the *principal value,* is understood by the equation $y=\sin^{-1}x$. Thus

$$-\frac{\pi}{2} \leqslant \sin^{-1}x \leqslant \frac{\pi}{2}$$

In the case of $y=\cos^{-1}x$, for any value of x between -1 to 1, there is only one value of y if y is restricted to the value between 0 and π which is taken as its principal value. Thus

$$0 \leqslant \cos^{-1}x \leqslant \pi$$

similarly $$-\frac{\pi}{2} \leqslant \tan^{-1}x \leqslant \frac{\pi}{2}$$

These principal values are shown by heavily marked branches of the graphs.

$\operatorname{Sec}^{-1}x$ and $\operatorname{cosec}^{-1}x$ are supposed to lie in the first or third quadrants.

Thus $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)=\frac{-\pi}{4}$, $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right))=\frac{3\pi}{4}$,

$\tan^{-1}(-1)=-\frac{\pi}{4}$, $\sec^{-1}(2)=\frac{\pi}{3}$, $\sec^{-1}-2\ )=4\pi/3$.

Note. The graphs of inverse trigonometric functions are obtained by reflecting graphs corresponding to direct functions in the line $y=x$. This can be seen by drawing the graph of $\sin x$ and reflecting it in the line $y=x$ in the same figure. The graph so obtained is that of $\sin^{-1}x$.

4·8. **Derivatives of the inverse trigonometric functions** are now obtained from first principles as follows :—

4·81. Differentiation of sin⁻¹x.

Let $y=\sin^{-1}x$. then $x=\sin y$ ....(1)

$\therefore\ y+\delta y=\sin^{-1}(x+\delta x)$, so that $x+\delta x=\sin(y+\delta y)$.

$$\therefore \qquad \delta x=\sin(y+\delta y)-\sin y$$

and $$\frac{\delta y}{\delta x}=\frac{\delta y}{\sin(y+\delta y)-\sin y}=\frac{\delta y}{2\cos(y+\frac{1}{2}\delta y)\sin\frac{1}{2}\delta y}.$$

Now when $\delta x\rightarrow 0$, δy also $\rightarrow 0$

$$\therefore\ \frac{dy}{dx}=\underset{\delta x\rightarrow 0}{\text{Lt}}\frac{\delta y}{\delta x}=\underset{\delta y\rightarrow 0}{\text{Lt}}\frac{1}{\cos(y+\frac{1}{2}\delta y)}\times\underset{\delta y\rightarrow 0}{\text{Lt}}\frac{\frac{1}{2}\delta y}{\sin\frac{1}{2}\delta y}$$

$$=\frac{1}{\cos y}\qquad\qquad\text{.........(2)}$$

Now as y is restricted to have values between $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$, cos y is positive and, therefore,

$$\cos y=+\sqrt{1-\sin^2 y}$$

$$\therefore\ \frac{dy}{dx}=\frac{1}{\cos y}=\frac{1}{\sqrt{1-\sin^2 y}}=\frac{1}{\sqrt{1-x^2}}\qquad[\text{by }(1)]$$

Hence $$\frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{\sin^{-1}x})=\frac{\mathbf{1}}{\sqrt{\mathbf{1-x^2}}}.$$

Note. The various steps on the right hand sides of the equations are the same as those required for differentiating sin y[as in Art. 4·1 for sin x]. If the derivative of sin y with respect to y which is cos y is taken for granted, the result (2) is briefly obtained by differentiating equation (1) with respect to x, which gives at once.

$$\frac{dx}{dy}=\cos y\text{ or }\frac{dy}{dx}=\frac{1}{\cos y}.$$

***4·811.** Geometrical. Let OA=OP=OQ=1

and let $y=\sin^{-1}x=\angle AOP=\text{arc AP}$.

$$y+\delta y=\sin^{-1}(x+\delta x)=\angle AOQ=\text{arc AQ}.$$

[See page 50]

$$\delta y=\sin^{-1}(x+\delta x)-\sin^{-1}x$$
$$=\text{arc AQ}-\text{arc AP}$$
$$=\text{arc PQ}.$$

Also $$\begin{aligned}\delta x&=(x+\delta x)-x\\&=\sin(y+\delta y)-\sin y\\&=\sin\angle AOQ-\sin\angle AOP=NQ-MP=RQ\end{aligned}$$

$$\frac{\delta y}{\delta x}=\frac{\text{arc } PQ}{RQ}=\frac{\text{arc } PQ}{\text{chord } PQ}\times\frac{\text{chord } PQ}{RQ}$$

$$\therefore\quad \frac{dy}{dx}=\text{Lt.}\frac{\text{arc } PQ}{\text{chord } PQ}\times\text{Lt.}\frac{\text{chord } PQ}{RQ}$$

$$=1.\quad \text{Lt. sec } \angle PQR$$

$$=\sec\ \angle AOP=\sec y=\frac{1}{\sqrt{1-\sin^2 y}}=\frac{1}{\sqrt{1-x^2}}$$

4·82. Differentiation of $\cos^{-1}x$

Let $y=\cos^{-1}x$, then $x=\cos y$ ...(1)

$y+\delta y=\cos^{-1}(x+\delta x)$, $\therefore x+\delta x=\cos(y+\delta y)$

So that $\delta x=\cos(y+\delta y)-\cos y$

$$\therefore\quad \frac{\delta y}{\delta x}=\frac{\delta y}{\cos(y+\delta y)-\cos y}=\frac{\delta y}{-2\sin(y+\frac{1}{2}\delta y)\sin\frac{1}{2}\delta y}$$

Now when $\delta x\to 0$, δy also $\to 0$,

$$\therefore\quad \frac{dy}{dx}=\underset{\delta y\to 0}{\text{Lt}}\frac{\delta y}{\delta x}$$

$$=-\underset{\delta y\to 0}{\text{Lt}}\frac{1}{\sin(y+\frac{1}{2}\delta y)}\times\underset{\delta y\to 0}{\text{Lt}}\frac{\frac{1}{2}\delta y}{\sin\frac{1}{2}\delta y}=-\frac{1}{\sin y}\quad\text{...(2)}$$

Now as y lies between o and π, $\sin y$ is positive and equal to $+\sqrt{1-\cos^2 y}$

$$\therefore\quad \frac{dy}{dx}=-\frac{1}{\sin y}=-\frac{1}{\sqrt{1-\cos^2 y}}$$

$$=-\frac{1}{\sqrt{1-x^2}}\qquad\text{[By (1)}$$

Hence $$\frac{d}{dx}(\cos^{-1}x)=-\frac{1}{\sqrt{1-x^2}}$$

Note. Note of Art 4·81 is applicable to this case also, so that if the derivative of cos y which is $-\sin y$ is assumed, result (2) can be obtained at once by differentiating equation (1) with respect to y, which gives $\frac{dx}{dy}=-\sin y$ or $\frac{dy}{dx}=-\frac{1}{\sin y}$.

*Ex. Differentiate $\cos^{-1}x$ geometrically.

4·83. Differentiation of $\tan^{-1}x$.

Let $y=\tan^{-1}x$, then $x=\tan y$..............(1)

$y+\delta y=\tan^{-1}(x+\delta x)$, $\therefore$ $x+\delta x=\tan(y+\delta y)$

$$\therefore\ \delta x=\tan(y+\delta y)-\tan y=\frac{\sin(y+\delta y)}{\cos(y+\delta y)}-\frac{\sin y}{\cos y}$$

$$=\frac{\sin\delta y}{\cos y\cos(y+\delta y)}$$

$$\therefore\ \frac{\delta y}{\delta x}=\delta y\times\frac{\cos y\cos(y+\delta y)}{\sin\delta y}=\frac{\delta y}{\sin\delta y}.\cos y\cos(y+\delta y)$$

Now when $\delta x\to 0$, δy also$\to 0$

$$\therefore\ \frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\frac{\delta y}{\delta x}$$

$$=\underset{\delta y\to 0}{\text{Lt}}\frac{\delta y}{\sin\delta y}.\ \underset{\delta y\to 0}{\text{Lt}}\cos y\cos(y+\delta y)=\cos^2 y\ldots(2)$$

$$\therefore\ \frac{dy}{dx}=\cos^2 y=\frac{1}{\sec^2 y}=\frac{1}{1+\tan^2 y}=\frac{1}{1+x^2}\qquad[\text{by }(1)]$$

Hence $\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$.

Note. If the derivative of tan y is assumed, the result (2) can be obtained at once by differentiating equation (1) with respect to x. which gives

$$\frac{dx}{dy}=\sec^2 y\text{ or }\frac{dy}{dx}=\cos^2 y.$$

***4·831. Geometrical.** Let OA=OP=OQ=1, and let $y=\tan^{-1}x=\angle \text{AOP}=\text{arc AP}$ (See page 52)

$$\begin{aligned} y+\delta y &= \tan^{-1}(x+\delta x) \\ &= \angle \text{AOQ} = \text{arc AQ} \\ \delta y &= \tan^{-1}(x+\delta x)-\tan^{-1}x \\ &= \text{arc AQ}-\text{arc AP}=\text{arc PQ} \end{aligned}$$

Also $\delta x=(x+\delta x)-x=\tan(y+\delta y)-\tan y$
$=\text{AC}-\text{AB}=\text{BC}.$

B and C being the points where OP, OQ meet the tangent at A and BN ⊥ OC,

$$\therefore \frac{\delta y}{\delta x}=\frac{\text{arc PQ}}{\text{BC}}=\frac{\text{BN}}{\text{BC}}\cdot\frac{\text{arc PQ}}{\text{BN}}$$

$$\therefore \frac{dy}{dx}=\text{Lt}\,\frac{\text{BN}}{\text{BC}}\times\text{Lt}\,\frac{\text{arc PQ}}{\text{BN}}=\text{Lt}\,\sin\angle\text{NCB}\times\text{Lt}\,\frac{\text{OP}}{\text{OB}}$$

$$=\sin\angle\text{OBA}\times\frac{\text{OA}}{\text{OB}}=\sin\left(\frac{\pi}{2}-y\right)\times\cos y$$

$$=\frac{1}{\sec^2 y}=\frac{1}{1+\tan^2 y}=\frac{1}{1+x^2}.$$

Example. 1. Differentiate $\sin^{-1}\sqrt{x}$.

Let $y=\sin^{-1}\sqrt{x}$ and $u=\sqrt{x}$ then $y=\sin^{-1}u$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{1}{\sqrt{1-u^2}}\times\frac{1}{2\sqrt{x}}=\frac{1}{2\sqrt{x}\sqrt{1-x}}$$

4·84. Differentiation of $\cot^{-1}x$

Let $y=\cot^{-1}x$, then $x=\cot y$ (1)

Differentiating equation (1) with respect to y, we have

$$\frac{dx}{dy}=-\text{cosec}^2 y \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\therefore \frac{dy}{dx}=-\frac{1}{\text{cosec}^2 y}=-\frac{1}{1+\cot^2 y}=-\frac{1}{1+x^2}\ \text{[By (1)]}$$

Note. Here the differential coefficient of cot y is assumed. In case the derivative is to be obtained from first principles, full process may be shown just as in Art. 4·83 for $\tan^{-1}x$.

This is left to the student as an exercise.

4·85. Derivative of $\sec^{-1}x$.

Let $y=\sec^{-1}x$, then $x=\sec y$..................(1)

$y+\delta y=\sec^{-1}(x+\delta x)$, $\therefore$ $x+\delta x=\sec(y+\delta y)$

$$\therefore \quad \delta x=\sec(y+\delta y)-\sec y=\frac{1}{\cos(y+\delta y)}-\frac{1}{\cos y}$$

$$=\frac{\cos y-\cos(y+\delta y)}{\cos y\cos(y+\delta y)}=\frac{2\sin(y+\frac{1}{2}\delta y)\sin\frac{1}{2}\delta y}{\cos y\cos(y+\delta y)}$$

$$\therefore \quad \frac{\delta y}{\delta x}=\delta y\times\frac{\cos y\cos(y+\delta y)}{2\sin(y+\frac{1}{2}\delta y)\sin\frac{1}{2}\delta y}.$$

$$\therefore \quad \frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\frac{\delta y}{\delta x}=\underset{\delta y\to 0}{\text{Lt}}\frac{\frac{1}{2}\delta y}{\sin\frac{1}{2}\delta y}\ \underset{\delta y\to 0}{\text{Lt.}}\frac{\cos y\cos(y+\delta y)}{\sin(y+\frac{1}{2}\delta y)}$$

$$=1\times\frac{\cos^2 y}{\sin y}=\frac{1}{\sin y\sec^2 y}=\frac{1}{\sec y\tan y}\ \text{...(2)}$$

Now y is supposed to lie between 0 and $\frac{1}{2}\pi$ or between π and $\frac{3}{2}\pi$, so that $\tan y$ is positive and equal to $+\sqrt{\sec^2 y-1}$

$$\therefore \quad \frac{dy}{dx}=\frac{1}{\sec y\sqrt{\sec^2 y-1}}=\frac{1}{x\sqrt{x^2-1}} \qquad \text{[by (1)]}$$

Hence $$\frac{d}{dx}(\sec^{-1}x)=\frac{1}{x\sqrt{x^2-1}}.$$

Note. On the assumption of the fact that the derivative of sec y is sec y tan y, result (2) can be written down at once by differentiating equation (1) with respect to y.

*Ex. Differentiate $\sec^{-1}x$ geometrically.

4·86. Differentiation of $\operatorname{cosec}^{-1}x$.

Let $y=\operatorname{cosec}^{-1}x$, then $x=\operatorname{cosec} y$....................(1)

Differentiating equation (1) with respect to y, we have

$$\frac{dx}{dy}=-\operatorname{cosec} y \cot y,$$

$$\therefore \quad \frac{dy}{dx}=-\frac{1}{\operatorname{cosec} y \cot y}. \qquad \ldots(2)$$

Now y is restricted to lie between 0 and $\frac{1}{2}\pi$ or between π and $\frac{3}{2}\pi$, so that $\cot y$ is positive and equal to $+\sqrt{\operatorname{cosec}^2 y-1}$.

$$\therefore \quad \frac{dy}{dx}=-\frac{1}{\operatorname{cosec} y\sqrt{\operatorname{cosec}^2 y-1}}=-\frac{1}{x\sqrt{x^2-1}} \quad \text{[by (1)]}$$

$$\text{Hence } \frac{d}{dx}(\operatorname{cosec}^{-1}x)=-\frac{1}{x\sqrt{x^2-1}}.$$

Note. Here the derivative of $\operatorname{cosec} y$ is assumed. In case the derivative is to be obtained from first principles full process may be shown just as in Art. 4·85 for $\sec^{-1}x$. This is left to the student as an exercise.

Ex. Differentiate (*i*) $\tan^{-1}\sqrt{x}$ (*ii*) $\cos^{-1}(1-2x)$ (*iii*) $x^2\sec^{-1}x$.

(*i*) Let $y=\tan^{-1}\sqrt{x}$ and $u=\sqrt{x}$, then $y=\tan^{-1}u$

$$\therefore \quad \frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{1}{1+u^2}\times\frac{1}{2\sqrt{x}}=\frac{1}{2\sqrt{x}(1+x)}$$

(*ii*) Let $y=\cos^{-1}(1-2x)$ and $u=1-2x$, then $y=\cos^{-1}u$.

$$\therefore \quad \frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=-\frac{1}{\sqrt{1-u^2}}(-2)=\frac{2}{\sqrt{1-(1-2x)^2}}$$

$$=\frac{1}{\sqrt{x-x^2}}.$$

(*iii*) Let $y=x^2\sec^{-1}x$, then

$$\frac{dy}{dx}=(2x)\sec^{-1}x+x^2\times\frac{1}{x\sqrt{x^2-1}}=2x\sec^{-1}x+\frac{x}{\sqrt{x^2-1}}$$

4·87 Sometimes a trigonometrical transformation considerably simplifies the process of differentiation.

To illustrate this, we solve the following examples in two ways.

Ex. 1. Find $\frac{dy}{dx}$ when $y=\tan^{-1}\left(\frac{1-\cos x}{\sin x}\right)$.

First Method. Let $u=\frac{1-\cos x}{\sin x}$ then $y=\tan^{-1}u$ and

$$\frac{dy}{du}=\frac{1}{1+u^2},\ \frac{du}{dx}=\frac{\sin x(\sin x)-\cos x(1-\cos x)}{\sin^2 x}=\frac{1-\cos x}{\sin^2 x}$$

also $1+u^2=1+\frac{(1-\cos x)^2}{\sin^2 x}=\frac{\sin^2 x+1-2\cos x+\cos^2 x}{\sin^2 x}$

$$=\frac{2(1-\cos x)}{\sin^2 x}.$$

$$\therefore\ \frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{1}{1+u^2}\cdot\frac{du}{dx}=\frac{\sin^2 x}{2(1-\cos x)}\times\frac{1-\cos x}{\sin^2 x}$$

$$=\tfrac{1}{2}.$$

Second Method. We know that

$$\frac{1-\cos x}{\sin x}=\frac{2\sin^2\frac{1}{2}x}{2\sin\frac{1}{2}x\cos\frac{1}{2}x}=\tan\frac{x}{2}.$$

$$\therefore\quad y=\tan^{-1}\frac{1-\cos x}{\sin x}=\tan^{-1}\left(\tan\frac{x}{2}\right)=\frac{x}{2}$$

$$\therefore\ \frac{dy}{dx}=\tfrac{1}{2}.$$

Ex. 2. if $y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$, find $\frac{dy}{dx}$.

First Method.

Let $u=2x/(1+x^2)$, then $y=\sin^{-1}u$.

$$\frac{dy}{du}=\frac{1}{\sqrt{1-u^2}}$$

and $$\frac{du}{dx}=2.\frac{1.(1+x^2)-x(2x)}{(1+x^2)^2}=\frac{2(1-x^2)}{(1+x^2)^2},$$

$$1-u^2=1-\frac{4x^2}{(1+x^2)^2}=\frac{(1-x^2)^2}{(1+x^2)^2}$$

$$\therefore\ \frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{1}{\sqrt{1-u^2}}\times\frac{du}{dx}$$

$$=\frac{1+x^2}{1-x^2}\times\frac{2(1-x^2)}{(1+x^2)^2}=\frac{2}{1+x^2}.$$

Second Method.

Put $x=\tan\theta$, then $\dfrac{2x}{1+x^2}=\dfrac{2\tan\theta}{1+\tan^2\theta}=\sin 2\theta.$

$\therefore\ y=\sin^{-1}(\sin 2\theta)=2\theta=2\tan^{-1}x \ \therefore\ \dfrac{dy}{dx}=\dfrac{2}{1+x^2},$

Note—The substitution is such that $2x/(1+x^2)$ becomes sine of some angle.

4.88. Formulae for Differentiation. The formulae derived in this chapter are collected in the following table for reference.

1. $\dfrac{d}{dx}(\sin x)=\cos x.$ 2. $\dfrac{d}{dx}(\cos x)=-\sin x$

3. $\dfrac{d}{dx}(\tan x)=\sec^2 x.$ 4. $\dfrac{d}{dx}(\cot x)=-\operatorname{cosec}^2 x.$

5. $\dfrac{d}{dx}(\sec x)=\sec x\tan x.$ 6. $\dfrac{d}{dx}(\operatorname{cosec} x)=-\operatorname{cosec} x\cot x$

7. $\dfrac{d}{dx}(\sin^{-1}x)=\dfrac{1}{\sqrt{1-x^2}}.$ 8. $\dfrac{d}{dx}(\cos^{-1}x)=-\dfrac{1}{\sqrt{1-x^2}}$

9. $\dfrac{d}{dx}(\tan^{-1}x)=\dfrac{1}{1+x^2}.$ 10. $\dfrac{d}{dx}(\cot^{-1}x)=-\dfrac{1}{1+x^2}$

11. $\dfrac{d}{dx}(\sec^{-1}x)=\dfrac{1}{x\sqrt{x^2-1}}.$

12. $\dfrac{d}{dx}(\operatorname{cosec}^{-1}x)=-\dfrac{1}{x\sqrt{x^2-1}}$

Exercise X

Find the derivative of each of the following functions with respect to the independent variable :

1. (*i*) $\sin^{-1}3x$. (*ii*) $\cos^{-1}(3x-1)$. (*iii*) $\tan^{-1}\left(\dfrac{1}{z}\right)$.

(*iv*) $\sec^{-1}\theta^2$ (*v*) $\sin^{-1}\sqrt{x}$ (*vi*) $\operatorname{cosec}^{-1}\sqrt{4x+1}$.

2. (*i*) $\operatorname{cosec}^{-1}(\sec x)$. (*ii*) $(1+x^2)\tan^{-1}x$.

(*iii*) $\cos^{-1}(\tan x^2)$. (P.U.1951) (*iv*) $\tan^{1}\left(\frac{}{1-x}\right)$. (*v*) $x \sec^{-1}x$.

(*vi*) $\sqrt{\sec^{-1}x^2}$. (P. U. 1950)

(*vii*) $(\tan^{-1} 2x)^3$ (*viii*) $\frac{1}{2}\tan^{-1}(\frac{1}{2}\tan\frac{1}{2}x)$.

3. (*i*) $x\sin^{-1}x+\sqrt{1-x^2}$. (*ii*) $x\cos^{-1}2x-\frac{1}{2}\sqrt{1-4x^2}$

(*iii*) $\frac{1}{2}x\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$.

(*iv*) $\tan^{-1}(\sin x+\cos x)$. (P. U. 1953)

(*v*) $\tan^{-1}(\sec x+\tan x)$.

(*vi*) $x\cos^{-1}x/\sqrt{1-x^2}$. (P. U. 1942)

4. (*i*) $\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$. (P. U. 1951)

(*ii*) $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$. (P. U. 1948)

[Hint. $\cos x/(1+\sin x)=\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)$]

5. (*i*) $\tan^{-1}\left(\frac{4\sin x}{3+5\cos x}\right)$. (*ii*) $\cos^{-1}\left(\frac{1+2\cos x}{2+\cos x}\right)$.

(*iii*) $\sin^{-1}\left(\frac{a+b\cos x}{b+a\cos x}\right)$. (P. U. 1950)

(*iv*) $\cos^{-1}\left(\frac{a+b\sin x}{b+a\sin x}\right)$. (P. U. 1953)

6. (*i*) $\sin^{-1}(2x\sqrt{1-x^2})$. (*ii*) $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$.

(*iii*) $\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.

(*iv*) $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$ (P. U. 1949

[Hint. Put $x=\sin\theta$ in (*i*) and $x=\tan\theta$ in others.]

7. (*i*) If $y=\frac{\sin^{-1}x}{\sqrt{1-x^2}}$, show that $(1-x^2)\frac{dy}{dx}-xy=1$

(P. U. 1953)

(*ii*) If $y=\sin(2\sin^{-1}x)$, prove that

$$\frac{dy}{dx}=2\sqrt{\frac{1-y^2}{1-x^2}}.$$

8. If s is the arc from the x-axis to the point (x, y) on the circle $x^2+y^2=a^2$, show that

$$s=a\cos^{-1}\frac{x}{a} \text{ and } \frac{ds}{dx}=-\frac{a}{y}.$$

9. Show that the derivatives of $\tan^{-1}\frac{x}{a}$ and $\tan^{-1}\frac{x-a}{x+a}$ are the same. Explain why it is so.

10. Show that $dy/dx=0$, when

(*i*) $y=\sin^{-1}x+\cos^{-1}x$ (*ii*) $y=\tan^{-1}x+\tan^{-1}\frac{1}{x}$.

(*iii*) $y=\sec^{-1}[(x+1)/(x-1)]+\sin^{-1}[(x-1)/(x+1)]$.

Explain why it is so.

11. Point out the fallacy in the following :

$$\frac{d}{dx}(\sin^{-1}x)=\frac{d}{dx}(-\cos^{-1}x) \qquad \therefore\ \sin^{-1}x+\cos^{-1}x=0$$

Putting $x=1$, we get $\frac{1}{2}\pi=0$ or $\pi=0$.

12. Prove the formula $\frac{dy}{dx}\times\frac{dx}{dy}=1$ and use it to find the derivative of $\sin^{-1}x$. (D. U.)

13. Find geometrically the differential coefficient of $\tan^{-1}x$. (D. U.)

14. Prove that (*i*) $\frac{d}{dx}\left[\sin^{-1}\frac{x}{a}\right]=\frac{a}{\sqrt{a^2-x^2}}$.

(*ii*) $\frac{d}{dx}\left[\tan^{-1}\frac{x}{a}\right]=\frac{a}{a^2+x^2}$.

(*iii*) $\frac{d}{dx}\left[\operatorname{cosec}^{-1}\frac{x}{a}\right]=-\frac{a}{x\sqrt{x^2-a^2}}$.

4·9. Derivatives of Higher Order. If y is a function of x then its differential coefficient is denoted by the symbol $\frac{dy}{dx}$ which is also generally a function of x and can therefore be again differentiated ; its differential co-efficient, *i.e.*, $\frac{d}{dx}\left[\frac{dy}{dx}\right]$ is denoted by the symbol $\frac{d^2y}{dx^2}$ and is the second differential co-efficient of y.

Similarly $\frac{d^3y}{dx^3}, \frac{d^4y}{dx^4} \cdots\cdots\cdots \frac{d^ny}{dx^n}$ are third, fourth...and nth derivatives of y ; the integers 3, 4, etc. denote the order of differentiation.

The successive differential co-efficients of y are also denoted by $y_1, y_2, y_3 \cdot y_n$ and if a function of x is denoted by $f(x)$ instead of y, its various derivatives are denoted by $f'(x)$, $f''(x) \ldots f^n(x)$. Another notation to denote the first, second, third...Differential Co-efficients of y is Dy, D^2y, D^3y......

Examples

Ex. 1. If $y=4x^3-3x^2+6x+7$, show that derivatives of the 4th and higher order all vanish.

$$y_1=12x^2-6x+6, \quad y_2=24x-6, \quad y_3=24,$$

$\therefore y_4=0$ and hence all derivatives of order higher than fourth vanish.

Ex. 2. If $y=A \sin nx+B \cos nx$, show that

$$\frac{d^2y}{dx^2}+n^2y=0$$

$$\frac{dy}{dx}=An \cos nx-Bn \sin nx=n(A \cos nx-B \sin nx)$$

$$\frac{d^2y}{dx^2}=n(-An \sin nx-Bn \cos nx)$$

$$=-n^2(A \sin nx+B \cos nx)=-n^2y$$

Hence $$\frac{d^2y}{dx^2}+n^2y=0$$

Ex. 3. Find the fourth derivative of
$$f(x)=(x^3-7)/(x+1).$$

$$f(x)=\frac{x^3-7}{x+1}=\frac{x^3+1}{x+1}-\frac{8}{x+1}=x^2-x+1-8(x+1)^{-1}$$
$$f'(x)=2x-1-8(-1)(x+1)^{-2}$$
$$f''(x)=2+8(-2)(x+1)^{-3}=2-16(x+1)^{-3}$$
$$f'''(x)=-16(-3)(x+1)^{-4}=48(x+1)^{-4}$$
$$f^{\text{iv}}(x)=48(-4)(x+1)^{-5}=-192(x+1)^{-6}$$

Ex. 4. If $y=\tan x$, prove that $\dfrac{d^2y}{dx^2}=2y\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}=2y(1+y^2)$ and find $\dfrac{d^3y}{dx^3}$ when $x=0$

Here $y=\tan x$ ......(1) $\therefore y_1=\sec^2x=1+\tan^2x$ ...(2)

$y_2=2\tan x\sec^2x=2\tan x(1+\tan^2x)$ ....(3)

From (1) and (3), we get,

$$\frac{d^2y}{dx^2}=2y(1+y^2) \quad \text{...(4)}$$

Also from (1), (2), and (3) we have

$$\frac{d^2y}{dx^2}=2\tan x(1+\tan^2x)=2y\frac{dy}{dx} \quad \text{.........(5)}$$

Again $y_2=2\tan x(1+\tan^2x)=2\tan x+2\tan^3x$

$$\therefore y_3=2\sec^2x+6\tan^2x\sec^2x=2\sec^2x(1+3\tan^2x)$$
$$=2(1+4\tan^2x+3\tan^4x)=2, \text{ when } x=0.$$

Exercise XI

1. If $f(x)=x^4$. show that $f''(2)=f''(-2)$ but that $f'''(2)$ and $f'''(-2)$ are not equal.
2. If $y=(1+x)/x$, find y_3 when $x=-1$.
3. Find $f''(x)$ when $f(x)=x\sqrt{2x+1}$.
4. Find the second and fourth differential co-efficients of :

(*i*) $\dfrac{1}{x+5}$. (*ii*) $\dfrac{x+3}{x-1}$. (*iii*) $\dfrac{x^2}{1+x}$.

(*iv*) $\cos 3x \cos 7x$. (*v*) $\cos^2 x$. (*vi*) $\sin^3 x$.
(*vii*) $x^4 \sin x$.

5. If $y=\frac{x}{\sqrt{1-x^2}}$, find $\frac{d^3y}{dx^3}$ (P. U. 1944)

6. Find $\frac{d^5y}{dx^5}$ when (*i*) $y=\frac{1}{(x-5)(x-6)}$
(*ii*) $y=\sin x \sin 3x$.

7. Prove that :

(*i*) If $y^2=4ax.\frac{d^2y}{dx^2}=-\frac{4a^2}{y^3}$.

(*ii*) If $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1; \frac{d^2y}{dx^2}=-\frac{b^4}{a^2y^3}$. [D. U.]

(*iii*) If $xy=ax^3+b$, $x^2\frac{d^2y}{dx^2}=2y$.

(*iv*) If $y=\frac{1}{x}(a \cos mx+b \sin mx)$,

$$\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}+m^2y=0.$$

8. (*i*) If $y=(\sin^{-1}x)^2$, show that $(1-x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}=2$.
(P. U. 1949)

(*ii*) If $xy=\sin x$, prove that $x\frac{d^2y}{dx^2}+2\frac{dy}{dx}+xy=0$.

(*iii*) If $p^2=a^2 \cos^2\theta+b^2 \sin^2\theta$, prove that

$$p+\frac{d^2p}{d\theta^2}=\frac{a^2b^2}{p^3}.$$ (P. U. 1949. S.)

9. (*a*) If $y=\frac{1}{x+1}$, show that

(*i*) $\frac{dy}{dx}. \frac{dx}{dy}$ and (*ii*) $\frac{d^2y}{dx^2}. \frac{d^2x}{dy^2}$

are both constant. Find the values of these constants.

(*b*) Are (*i*) and (*ii*) constant for all functions ?

(*c*) At what points does $\frac{d^2y}{dx^2}=\frac{d^2x}{dy^2}$?

10. If $=a+bx+cx^2$, show that

$$\frac{d^2y}{dt^2}\div\left(\frac{dx}{dt}\right)^3$$ is constant. (D. U.)

Revision Questions—I

1. Find the limit of $\frac{x^n-a^n}{x-a}$ when x approaches a. (P. U.)

2. Explain clearly what you understand by a differential co-efficient. Find from first principles the differential co-efficient of $\tan x$. (B. U.)

3. Find from definition the differential co-efficients of :—

(*a*) (*i*) x^{1}. (*ii*) $\sqrt{\sin x}$. (P. U. 1953)

(*b*) (*i*) $\operatorname{cosec} x$. (*ii*) $\cos^2 x$. (*iii*) $\sin^3 x$

(*c*) (*i*) $\frac{1}{\sqrt{x}}$. (*ii*) $\cos \frac{1}{2}x$. (P. U. 1935)

(*iii*) $\tan x$. (P. U. 1948)

4. Starting from the first principles obtain the differential co-efficient of $\frac{u}{v}$, where u and v are functions of x. (P. U. 1947)

5. Differentiate :—

(*i*) $\frac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}$. (*ii*) $\frac{\sec x+\tan x}{\sec x-\tan x}$

(*iii*) $\frac{3\tan x-\tan^3 x}{1-3\tan^2 x}$.

[Hint—Simplify the fractions before differentiating them.]

6. **Differentiate:—**

(*a*) (*i*) $x^2y+y^2x=a^2$ (P. U. 1917)
(*ii*) $\cos^{-1}(\tan x^2)$ (P. U. 1951)

(b) (i) $\sqrt{\sin\sqrt{1+x^2}}$

(ii) $\cos^{-1}\left(\dfrac{b+a\cos x}{a+b\cos x}\right)$ (P. U. 1949 S.)

(c) (i) $\dfrac{x}{a-\sqrt{a^2-x^2}}$ (ii) $\sqrt{\dfrac{1-\cos x}{1+\cos x}}$

(iii) $\dfrac{1+\tan x}{1-\tan x}$. (P. U.)

7. Differentiate with respect to x,

$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ and verify the proposition

$\dfrac{dy}{dx}=\dfrac{dy}{dt}\cdot\dfrac{dt}{dx}$ by putting $x=a\cos t$, $y=b\sin t$.

(P.U.)

8. Find from the definition $\dfrac{dy}{dx}$ when $y=\dfrac{ax+b}{cx+d}$

(P. U.)

Also prove that $2y_1 y_3=3y_2^2$.

9. If $\sin y=x\sin(a+y)$, prove that $\dfrac{dy}{dx}=\dfrac{\sin^2(a+y)}{\sin a}$

(P. U.)

10. If $x^3+y^3-3axy=0$, prove that :

(i) $\dfrac{dy}{dx}=0$ when $x=a.\sqrt[3]{2}$, $y=a.\sqrt[3]{4}$

(ii) $\dfrac{dx}{dy}=0$ when $x=a.\sqrt[3]{4}$, $y=a.\sqrt[3]{2}$

(P. U.)

CHAPTER V

TANGENTS AND NORMALS

5·1 The graph of a function. If two variables are so related that the value of one depends on the value of the other, the nature of the relationship may be represented by a curve or *graph.*

In drawing the graph of any function of x, say $y=f(x)$, we give different values to the independent variable x and obtain the corresponding values of y. Regarding these values of x and y as abscissae and ordinates respectively, we plot as many points as desirable and then draw a smooth curve through them. The curve so obtained is called the graph of the equation $y=f(x)$ or the graph of the function $f(x)$.

From his knowledge of Trigonometry and Analytical Geometry, the student is already familiar with the nature of the graphs of functions such as $y=\sin x$, $y=\tan x$, $y^2=4ax$, $x^2/a^2+y^2/b^2=1$ and $x^2/a^2-y^2/b^2=1$.

5·2 Continuous and Discontinuous functions. A function $y=f(x)$ is said to be continuous at $x=a$, if a small change in the value of x produces only a corresponding small change in the value of y *i.e.*, the value of $f(x)$ does not take a sudden jump when the change in x is infinitesimally small, so that when $x\to a$ from either side of the point $x=a$, $y\to f(a)$. In other words,

$$\underset{x\to a}{Lt}\ f(x)=f(a)$$

This implies that:

(a) *The function $f(x)$ is defined at $x=a$. i.e. $f(a)$ exists.*

(b) $\underset{x\to a}{\text{Lt}}\ f(x)$ *exists i.e. the function approaches a finite limit as x approaches the value a from either side of the point $x=a$.*

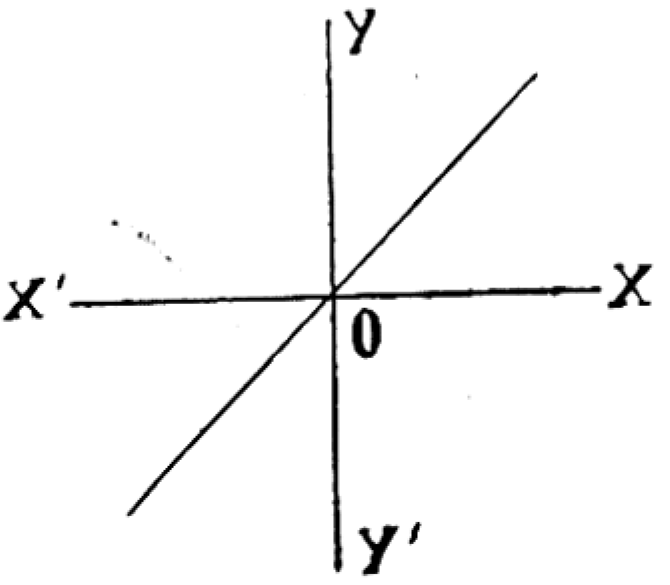
(c) *The limit approached is equal to the value of the function at the point i.e.* $\underset{x \to a}{Lt} f(x)=f(a)$.

If any of these conditions is not satisfied for $x=a$, the function is said to be *discontinuous* at that point.

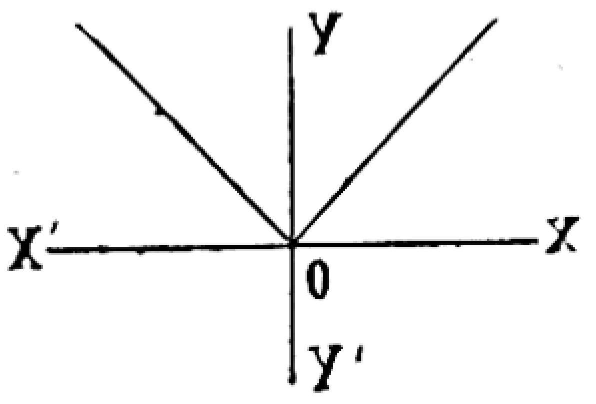
A function $f(x)$ is said to be continuous between the value $x=a$ and $x=b$ when it is continuous at every point between $x=a$ and $x=b$.

A continuous function can be represented, by a continuous and uninterrupted curve.

Examples



$y=x$

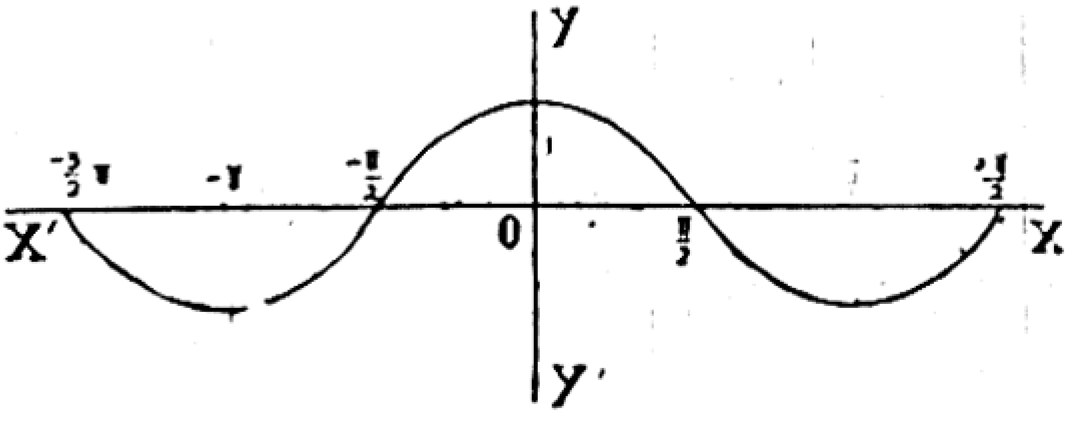


$y=+\sqrt{x^2}$

Ex. 1. $\mathbf{y=x}$ and $\mathbf{y=+\sqrt{x^2}}$

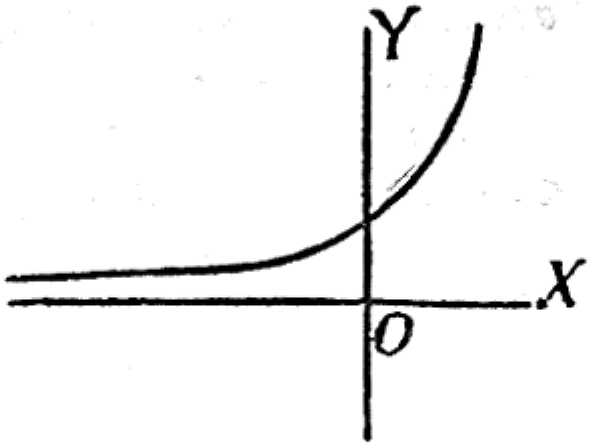
Both the curves are continuous everywhere. For $y=x$, y has the same sign as that of x, positive when x is positive and negative when x is negative but for $y=+\sqrt{x^2}$, y is always positive, as x^2 is positive even when x is negative.

Ex. 3. $\mathbf{y=\cos x}$.



x	$-\frac{3}{2}\pi$	$-\pi$	$-\frac{1}{2}\pi$	0	$\frac{1}{2}\pi$	π	$\frac{1}{2}\pi$
y	-0	-1	-0	1	0	-1	0

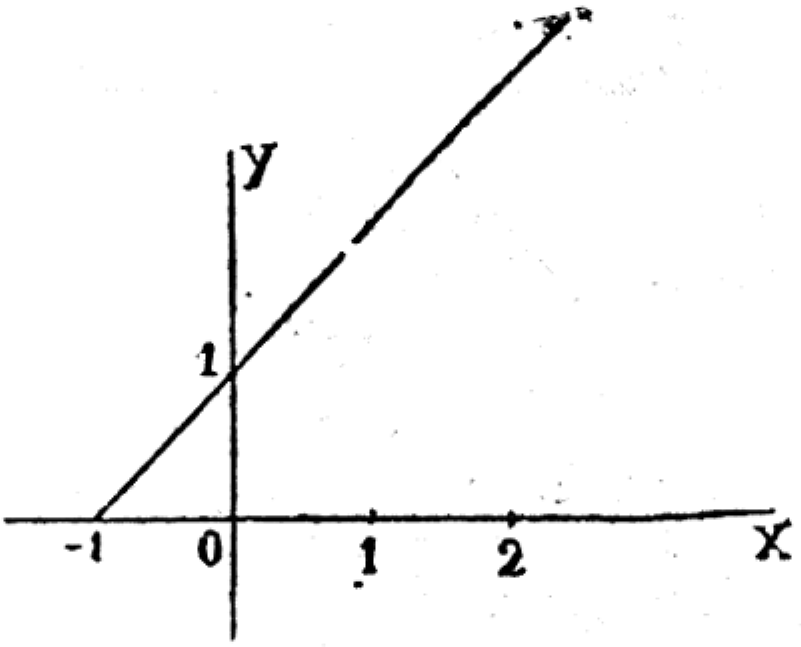
The curve is continuous throughout and can be extended both sides without break or jump.



Ex. 3. $y=3^x$.

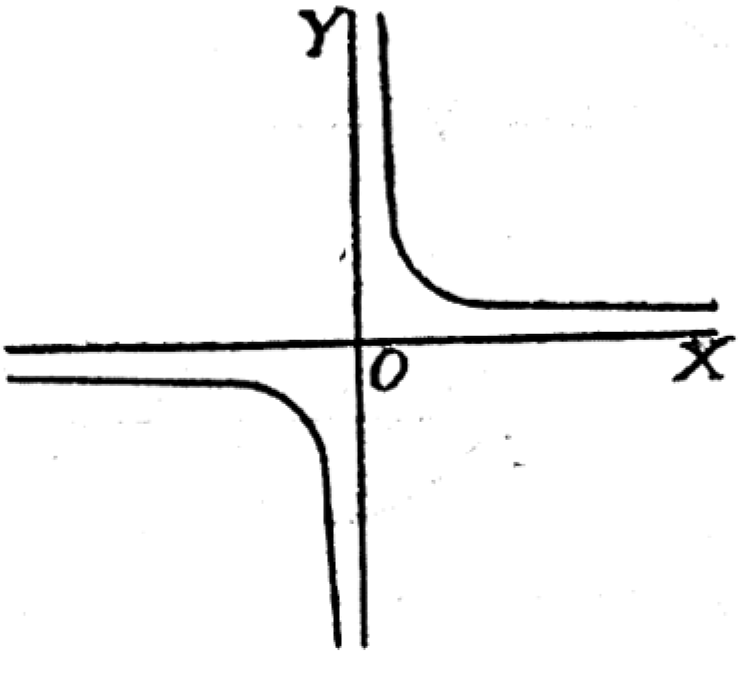
x	$\to-\infty$	0	1	$\to\infty$
y	$\to0$	1	3	$\to\infty$

3^x is a continuous function and is always positive.



Ex 4. $y=\frac{x^2-1}{x-1}$.

For all values of x, $x\neq1$, $y=x+1$, and the graph is then a straight line as shown in the figure. But when $x=1$, $y=\frac{0}{0}$ and has no definite value, so that y is not defined at $x=1$ and thus the curve is discontinuous at the point $x=1$.



Ex. 5. $y=\frac{1}{x}$

$x\to$	$-\infty$	-1	-0	$+0$	1	∞
$y\to$	-0	-1	$-\infty$	$+\infty$	1	0

The curve is continuous throughout except at the point $x=0$ where there is an infinite jump. As x approaches zero from the left hand side, y remains negative and becomes larger and larger in magnitude so that it tends to $-\infty$ but when y approaches zero from the right hand side, y is positive and again assumes larger and larger values so that it tends to $+\infty$ as $x \to 0$. Thus the curve jumps for $-\infty$ to $+\infty$ and is discontinuous at the point $x=0$.

There is another reason why $1/x$ is not continuous at $x=0$. It *does not exist when* $x=0$.

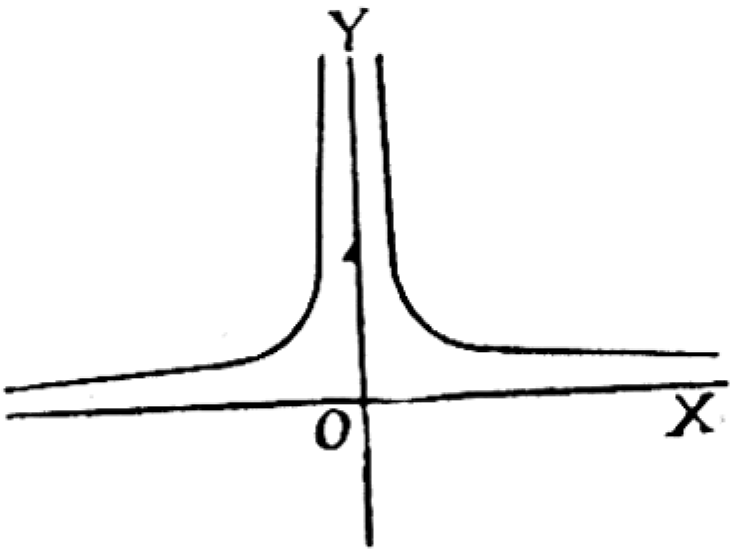
x and y are either both negative or both positive.

Ex. 6. $y=\dfrac{1}{x^2}$.

$x\to$	$-\propto$	-1	-0	$+0$	1	$+8$
$y\to$	$+0$	$+1$	$+\propto$	$+\infty$	1	0

The graph does not indicate a jump at $x=0$, but the function does not exist at that point and hence it is discontinuous at $x=0$.

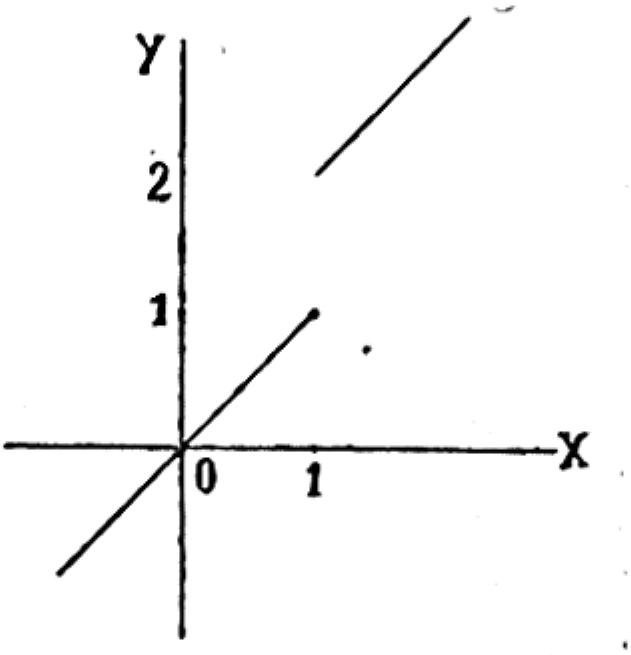
y is always positive, x^2 being positive for even negative values of x.



Ex. 7. $y=x$ when $x \leqslant 1$ and $y=x+1$ when $x>1$

The function is defined by two equations and is continuous throguhout except at the point $x=1$.

So long as $x<1$ or $=1$, the function is defined by the relation $y=x$ so that $y=1$ when $x=1$. But when $x=1+h$ where h is positive, the function is defined by the relation $y=x+1$ and thus $y=(1+h)+1$, so that



as $h \to 0$ $y \to 2$. But however small h may be taken $y \neq 2$ when $x=1$, for it is already defined to have the value 1 when $x=1$, the point (1,1) being on the first part of the line.

Ex. 8. Draw the graphs of the functions :—

(*i*) x^2 (*ii*) $\sqrt{x}$ (*iii*) $\sin x$ (*iv*) x^3 and show that they are continous functions.

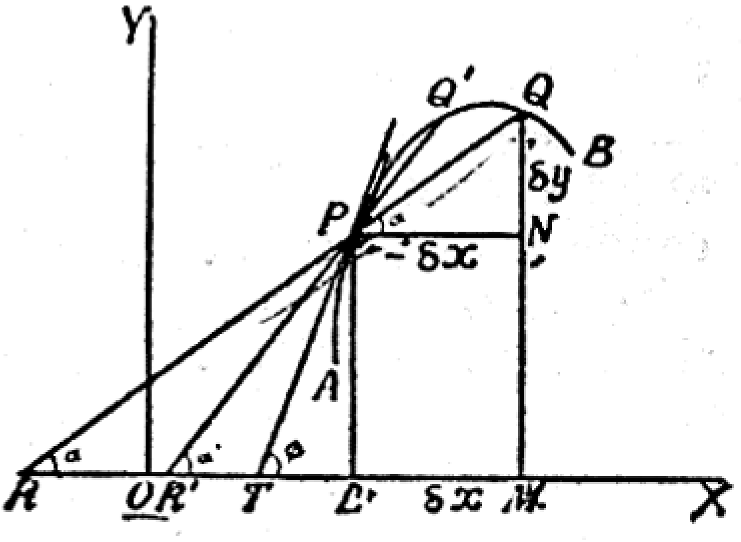
Ex. 9. Draw the graphs of :—

(*i*) $\frac{1}{x^3}$. (*ii*) $\frac{1}{x^4}$. (*iii*) $\frac{1}{x-1}$. (*iv*) $\frac{1}{(x-2)^2}$. (*v*) $\tan x$.

(*vi*) $\sec x$.

At what points are these functions discontinuous ?

5·3. Geometric Interpretation of the Derivative.



Let APQB represent the graph of a continuous function $y=f(x)$ and let $P(x,y)$ and $Q(x+\delta x, y+\delta y)$ be two points on it. Draw LP, MQ perpendiculars on the X-axis and $PN \perp MQ$, then

$$y=LP, \ y+\delta y=MQ$$

and $\delta y = MQ - LP$

$$= MQ - MN = NQ.$$

$$\therefore \ \frac{\delta y}{\delta x} = \frac{NQ}{PN} = \tan \angle QPN = \tan \angle QRX.$$

$$= \tan \alpha, \text{ slope of the secant PQ.}$$

Now let the point Q move along the curve towards the point P, so that secant PQ rotates about P and δx and δy become smaller and smaller.

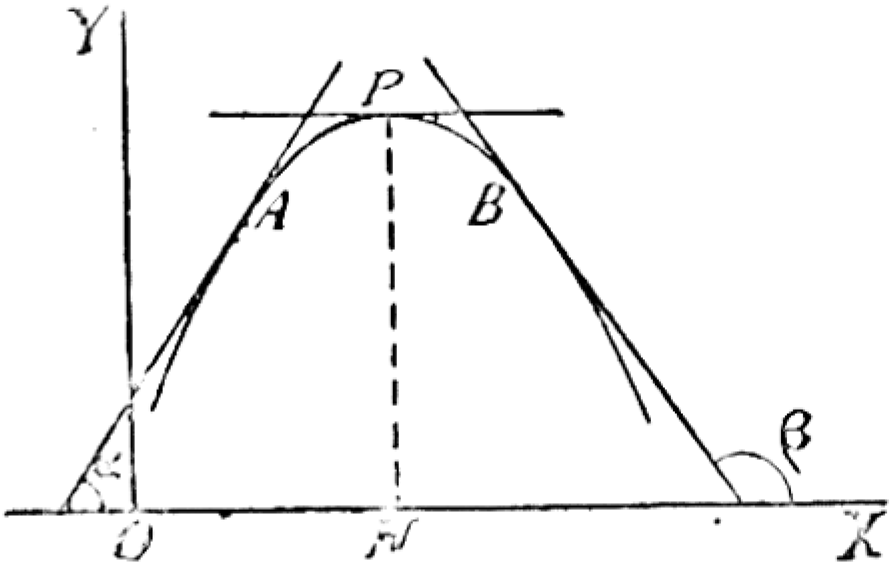
If Q′ be a subsequent position of the point Q, then $\frac{\delta y}{\delta x}$ = the slope of the secant PQ′ $= \tan \alpha'$. Ultimately when $\delta x \to 0$, and the point Q coincides with P, the secant PQ attains its limiting position PT, called *the tangent to the curve at P* and

$\frac{\delta y}{\delta x}$ approaches the limit $\frac{dy}{dx}=\tan\phi$, the **slope of the tangent** at P *i.e.*, the tangent of the angle which the tangent line at P makes with the axis of x.

But the slope the tangent line at P is called the *slope of the curve* at P.

Thus **the derivative of a function at a point is the slope of the curve at that point.**

5·31 The sign of the Derivative. As the tangent of an acute angle is positive and that of an obtuse angle negative, the value of $\frac{dy}{dx}$ for a particular point on a curve is positive or negative according as the tangent line makes an acute angle (as at A in the figure) or an obtuse angle (as at B) with the axis of x.



When the tangent at any point of a curve is parallel to the axis of x, $\frac{dy}{dx}=0$ at that point as at P in the figure.

For example, (*i*) if $y=x^2, \frac{dy}{dx}=2x$ which is zero at the origin where $x=0$. See Fig. on P. 22, the axis of x itself is tangent to the curve at O.

(*ii*) if $y=\cos x$, $\frac{dy}{dx}=-\sin x$, which is zero at the points $x=-\pi$, $x=0$, $x=\pi$ etc , as at points A, B, C of the figure Ex. 2, Art. 5·2.

Examples

Ex. 1. Find the points on the curve $y=x^4-2x^2$ where the tangent is parallel to the x-axis.

We have, $y=x^4-2x^2$, $\dfrac{dy}{dx}=4x^3-4x=4x(x-1)(x+1)$

This equated to zero gives $x=0$, $x=1$ and $x=-1$.

The corresponding values of y are $y=0$, $y=-1$ and $y=-1$.

Hence the tangents are parallal to the x-axis at points $(0, 0)$, $(1, -1)$ and $(-1, -1)$.

Ex. 2. At what point on the curve $y=x^2-2x$ does the tangent make an angle of 45° with the x-axis ?

Also find the point where the tangent is (*i*) parallel (*ii*) perpendicular to the line $y=2x+3$.

We have, $y=x^2-2x$ $\dfrac{dy}{dx}=2x-2$.

The point where the tangent makes an angle of 45° with the x-axis, the slope of the tangent$=\tan 45°=1$.

$\therefore \quad 2x-2=1 \qquad$ *i.e.*, $\quad x=\frac{3}{2}$.

Substituting this in the equation of the curve, we have $\quad y=\frac{9}{4}-2\times\frac{3}{2}=\frac{9}{4}-3=-\frac{3}{4}$,

Thus the required point is $(\frac{3}{2}, -\frac{3}{4})$.

In the second case, (*i*) the tangent is *parallel* to the given line if its slope which is $2x-2$ is equal to the slope of the given line which is 2,

$\therefore \quad 2x-2=2$ or $x=2$ and *i.e.*, $y=4-4=0$.

Thus the point is $(2, 0)$

(*ii*) The tangent shall be ***perpendicular*** to the given line if its slope$=-\frac{1}{2}$.

$\therefore \quad 2x-2=-\frac{1}{2}$ or $x=\frac{3}{4}$ which gives $y=\frac{9}{16}-2\times\frac{3}{4}$
$$=-\frac{15}{16}$$

Hence the required point is $(\frac{3}{4}, -\frac{15}{16})$.

Exercise XII

1. Find the slope of the curve at the given point :—
(*i*) $y=2x^3+1$ at $(-2, -15)$ (*ii*) $y^2=1+2x$ at $(4, 3)$
(*iii*) $y=\dfrac{1}{x-1}$ at $(2, 1)$ (*iv*) $y=\dfrac{1}{x^2}$ at $(-3, \frac{1}{9})$.

2 Find the slope of the curve $y=x^2$ and the point where the tangent is (*i*) parallel (*ii*) perpendicular to the line $y=4x+7$.

3. For what points on the curve $y=2x^3-6x-5$, is the tangent parallel to x-axis and at what points is it perpendicular to the line $x+18y=0$?

4. Find the angle at which the curve $y=\dfrac{x}{1+x}$ cuts the axis of x.

(P. U.)

5. At what points of the curve $y=\dfrac{x^2+5}{x-2}$ is the tangent parallel to the x-axis ?

Find the slope of the curve at the point $(1, -6)$.

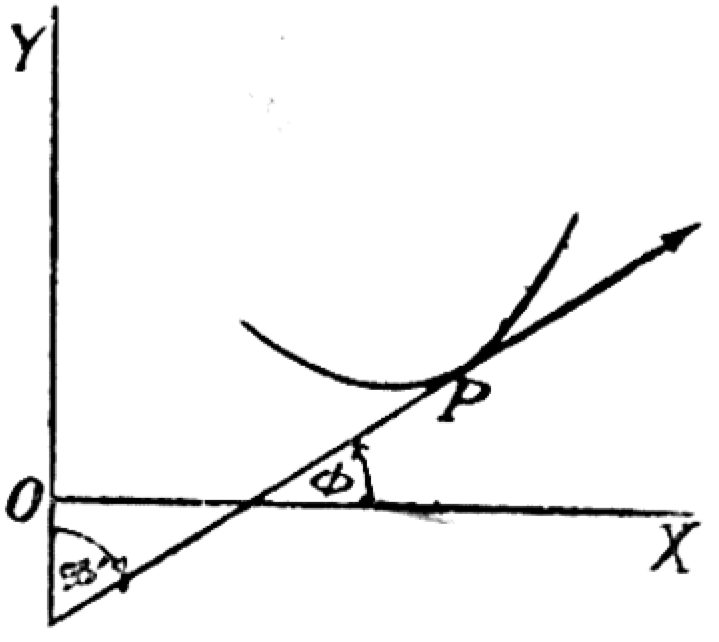
6. Find the co-ordinates of the points where the tangents to the circle $x^2+y^2=a^2$ are parallel to the x-axis and y-axis.

Also find the points, where the tangent is parallel to the line $x-\sqrt{3}\,y+7=0$.

7. Find the points where the tangents to the ellipse $4x^2+9y^2=1$ are (*i*) parallel (*ii*) perpendicular to $2y+x=0$.

8. Interpret geometrically the fact that the derivative of a constant is zero.

5·32. The *sense of the tangent* at a point on a curve is that in which x increases, as shown by the arrow in the figure, and the angle ϕ which it makes with the x-axis is measured in the anti-clockwise direction from OX towards OY.



5·33. Geometrical meaning of $\frac{dx}{dy}$. Taking the direction of the tangent as shown in the figure, the angle ϕ' which it makes with the y-axis is measured in the clock-wise direction from OY towards OX.

$$\text{Now } \tan \phi' = \tan (90° - \phi) = \cot \phi = \frac{1}{\tan \phi}$$

$$= \frac{1}{\frac{dy}{dx}} = \frac{dx}{dy}.$$

Thus $\frac{dx}{dy}$ *is the tangent of the angle which the tangent line at any paint of a curve makes with the axis of* y.

Also since $\tan \phi' . \tan \phi = 1$

$$\therefore \frac{dx}{dy} \times \frac{dy}{dx} = 1,$$

a well known result interpreted geometrically.

Ex. Prove that $\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$ and give a geometrical interpretation of each of the differential coefficients occurring in the left-hand side of that relation. (B.U.)

5·4. Equations of Tangent and Normal. As explained in Art. 5·3, the slope m of the tangent to a curve $y = f(x)$ at a point P (x', y') is the same as the slope of the curve at that point and is equal to the value of $\frac{dy}{dx}$ or $f'(x)$ for that point, so that

$$m = f'(x') = \frac{dy'}{dx'},$$

where $\frac{dy'}{dx'}$ or $f'(x')$ stands for the value of the derivative of the curve for the point (x', y').

Now as the tangent to a curve at a point $P(x', y')$ on a curve is the line which passes through P and which has the same slope as the curve at P, therefore, its equation must be of the form

$$y-y'=m(x-x')$$

or $$y-y'=\frac{dy'}{dx'}(x-x') \qquad \text{......(1)}$$

or $$y-y'=f'(x')(x-x').$$

The Normal at $P(x', y')$ is a straight line which passes through P and is perpendicular to the tangent at P.

The normal being perpendicular to the tangent, *its slope is negative reciprocal of the slope of the tangent.*

$\because$ the slope of the tangent at P is $m=\frac{dy'}{dx'}$,

$\therefore$ the slope of the normal at P is $-\frac{1}{m}=-\frac{dx'}{dy'}$.

Hence the equation of the normal is

$$y-y'=-\frac{dx'}{dy'}(x-x')$$

or $$x-x'+\frac{dy'}{dx'}(y-y')=0 \qquad \text{......(2)}$$

Examples

Ex. 1. Find the equations of the tangents and normals to the circle $x^2+y^2=25$ at the points (x', y') and $(4, 3)$.

We have, $$x^2+y^2=25 \qquad \text{......(1)}$$

$\therefore$ $$2x+2y\frac{dy}{dx}=0$$

i.e., $\frac{dy}{dx}=-\frac{x}{y}=-\frac{x'}{y'}$ at (x', y').

$\therefore$ the equation of the tangent at (x', y') is

$$y-y'=-\frac{x'}{y'}(x-x') \quad \text{or} \quad yy'+xx'=x'^2+y'^2$$

Since (x',y') is on the circle (1), $\therefore\ x'^2+y'^2=25$, the equation of the tangent is, therefore, reduced to the form

$$xx'+yy'=25 \qquad \ldots\ldots(2)$$

At the point (4, 3) it becomes

$$4x+3y=25.$$

The normal at (x', y'), being perpendicular to the tangent, has the slope y'/x', and therefore, its equation is

$$y-y' = \frac{y'}{x'}(x-x') \qquad \text{or } x'y=xy' \ldots\ldots(3)$$

At the point (4, 3), it becomes $4y=3x$.

Ex. 2. Find the equation of the tangent and normal to the ellipse $x^2+2y^2 = 19$ at the point (1, 3).

We have, $x^2+2y^2 = 19$ and $2x+4y\dfrac{dy}{dx}=0$,

$\therefore$ the slope of the curve is $\dfrac{dy}{dx}=-\dfrac{x}{2y}=-\frac{1}{6}$ at (1, 3).

$\therefore$ the equation of the tangent at (1, 3) is

$$y-3=-\tfrac{1}{6}(x-1) \text{ or } x+6y=19.$$

The slope of the normal at (1, 3) is 6 and therefore its equation is $y-3=6(x-1)$ or $6x-y=3$.

5·41 In the above examples we have taken curves of the second degree in x and y which are usually dealt with in Co-ordinate Geometry. But the calculus method has the distinct advantage of being simpler and more general. We now illustrate this by taking a few examples of other curves.

Ex. 1. Find the equation of the tangent and normal to the curve $y=x^3-2x$ at the point $(1,-1)$.

Find also the equation of the tangent at the point where the tangent at $(1,-1)$ intersects the curve again.

We have, $\quad y=x^3-2x \quad \ldots\ \ldots\ \ldots \quad (1),$

and $\quad \dfrac{dy}{dx}=3x^2-2 \ldots\ \ldots\ \ldots \quad (2)$

m, the slope of the tangent at $(1,-1)$

$$=\frac{dy}{dx} \text{ at } (1,-1)=3-2=1$$

and the slope of the $\text{normal}=-\frac{1}{m}=-1$.

$\therefore$ the equation of the tangent at $(1,-1)$ is
$y+1=1\,(x-1)$ or $y=x-2$ ... ... (3)

and the equation of the normal at $(1,-1)$ is
$y+1=-1\,(x-1)$ or $y+x=0$ ... ... (4)

Now the abscissae of the points where the tangent $y=x-2$ intersects the curve $y=x^3-2x$ are given by the equation

$x-2=x^3-2x$ or $x^3-3x+2=0$ or $(x-1)^2(x+2)=0$

$\therefore$ either $x=1$ which gives the given point $(1,-1)$.
or $x=-2$ which gives the point $(-2,-4)$.
For $x=-2$, the slope$=10$ [by (2)].

$\therefore$ the equation of the tangent at $(-2,-4)$ is
$y+4=10\,(x+2)$ or $10x-y+16=0$.

Ex. 2. Find the equations of the tangent and normal to the curve $y=\sin x$ at the point where $x=\pi/3$.

We have, $y=\sin x$, $\therefore$ $\frac{dy}{dx}=\cos x=\frac{1}{2}$ for $x=\frac{\pi}{3}$.

By the equation of the curve, we have $y=\sqrt{3}/2$ when $x=\pi/3$.

$\therefore$ the equation of the tangent is

$$y-\frac{\sqrt{3}}{2}=\tfrac{1}{2}(x-\tfrac{1}{3}\pi) \quad\text{or}\quad 2y-x=\sqrt{3}-\pi/3.$$

slope of the normal at $x=\pi/3$ is -2,

$\therefore$ the equation of the normal is

$$y-\frac{\sqrt{3}}{2}=-2\,(x-\tfrac{1}{3}\pi) \quad\text{or}\quad 2y+4x=\sqrt{3}+4\pi/3.$$

Ex. 3. Find the equations of the tangent and normal to the curve $y\ \ x^2/(x-1)$ at the point whose abscissa is 3.

When $x=3$, $y=\frac{9}{2}$, $\therefore$ the point, is $(3, \frac{9}{2})$.

$$\frac{dy}{dx}=\frac{2x\,(x-1)-x^2}{(x-1)^2}=\frac{x^2-2x}{(x-1)^2}=\tfrac{3}{4}\text{ for }x=3,$$

$\therefore$ the equation of the tangent at the point $(3,\ \frac{9}{2})$ is $y-\frac{9}{2}=\frac{3}{4}(x-3)$ or $4y=3x+9$.

The slope of the normal $=-\frac{4}{3}$ and its equation is $y-\frac{9}{2}=-\frac{4}{3}(x-3)$ or $8x+6y=51$.

Exercise XIII

1. Find the equations of the tangent and normal to each of the following curves at the point indicated :

(*i*) $x^2+y^2=5$; $(1,-2)$ (*ii*) $y^2=-2x$; $(-2,\ 2)$
(*iii*) $x^2-y^2=16$; $(5,\ 3)$ (*iv*) $9x^2+16y^2=25$; $(1,1)$
(*v*) $xy=-6$; $(2,-3)$ (*vi*) $y^3=x^2$; $(-1,\ 1)$
(*vii*) $8y=x^3-12x+16$; $(0,\ 2)$
(*viii*) $y=2x^3+3x^2-1$; $(-1,\ 0)$
(*ix*) $4x^2+9y^2+18y-16=0$; $(2,\ 0)$
(*x*) $y=\frac{1}{2}\sin 2x$ at $x=\frac{1}{2}\pi$.

✓ **2.** Find the equation of the tangent to $6y=x^2(x-1)$ at the point where $x=-1$. Find also the equation of the tangent where this tangent meets the curve again.

3. Find the equation of the line which is

(*i*) tangent to $y^2=2x$ and parallel to $y=x$;
(*ii*) tangent to $x^2+y^2=4$ and perpendicular to $x-y+1=0$;
(*iii*) tangent to $y=2x^3-6x^2-2x+1$ and parallel to $2x+y=12$;
(*iv*) tangent to $y=\frac{1}{2}\left(x+\frac{1}{x}\right)$ and parallel to $2y+3x=0$

4. Find the equations of the normals to the curve $2x^2-3x+2y^2-xy=0$ at the point where $x=1$.
(P. U. 1946)

5. Find the equations of the tangent and normal to the following curves at the given point :

(*i*) $y=\dfrac{(x^2+5)}{x-2}$; $(1,\ -6)$ (*ii*) $y^2=\dfrac{8}{1+x^2}$; $(1,\ 2)$

(*iii*) $y=\frac{ax^2}{x^2+a^2}$ for $y=\frac{a}{4}$.

6. Find the equations of the tangents to the curves given in Q. 1 at the point (x', y').

7. Find the equations of the tangents to the curve $y=2x-x^2$ at $(0, 0)$, $(2, 0)$ and show that they meet on the line $x=1$.

8. Find the equations of the tangents parallel to the co-ordinate axes for the curve $x^2-2xy+3y^2=32$.

9. Find the equations of the tangents at the two points on the curve $x^2y^2=x^5+8y$ at which $x=2$.

5 42. Tangent for Parametric Equations of a curve. Let the equation of a curve be given in the form

$$x=f(t),\ y=\phi(t),$$

where x and y are expressed in terms of a third variable called a *parameter*.

In this case, since $\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}$,

we have, $$\frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}=\frac{\phi'(t)}{f'(t)},$$

and the equation of the tangent is

$$y-\phi(t)=\frac{\phi'(t)}{f'(t)}[x-f(t)].$$

Example. Find the equations of the tangent and normal to the ellipse $x=a\cos\theta$, $y=b\sin\theta$ at the point $\theta=\theta_1$.

Here $\frac{dx}{d\theta}=-a\sin\theta$, $\frac{dy}{d\theta}=b\cos\theta$.

$\therefore$ Slope of the tangent$=\frac{dy}{dx}=\frac{b\cos\theta}{-a\sin\theta}=-\frac{b\cos\theta_1}{a\sin\theta_1}$

at $\theta=\theta$

$\therefore$ the equation of the tangent is

$$y-b\sin\theta_1=-\frac{b\cos\theta_1}{a\sin\theta_1}(x-a\cos\theta_1)$$

or $bx\cos\theta_1+ay\sin\theta_1=ab(\cos^2\theta_1+\sin^2\theta_1)$

or $bx \cos \theta_1 + ay \sin \theta_1 = ab$

The slope of the normal is $a \sin \theta_1 / b \cos \theta_1$

$\therefore$ the equation of the normal is

$$y - b \sin \theta_1 = \frac{a \sin \theta_1}{b \cos \theta_1}(x - a \cos \theta_1).$$

Multiplying by b cosec θ_1 and tranposing, we get

$$ax \sec \theta_1 - by \operatorname{cosec} \theta_1 = a^2 - b^2.$$

5·43. Tangent at the origin. The equation of a straight line through the origin, other than the axis of y, is of the form $y = mx$, so that we substitute mx, for y in the equation of the curve and find the value of m in the limit when x and y both tend to zero.

Ex. 1. Find the equation of the tangent at the origin to $2y = 3x + 4x^3 + 5y^2$.

Putting $y = mx$, we get

$2mx = 3x + 4x^3 + 5m^2x^2$ or $2m = 3 + 4x^2 + 5m^2x$

Now when $x \to 0$, $2m = 3$,

$\therefore$ the equation of the required tangent is

$$y = \tfrac{3}{2}x \qquad \text{or} \qquad 2y = 3x.$$

This shows that the *equation of the tangent at the origin can be written down by suppressing the terms of degree higher than the first in the equation of the curve.*

Note. In this case the equation of the tangent can also be obtained by the usual method of finding the value of dy/dx for the origin.

Ex. 2. Find the equations of the tangents at the origin to the curve

$$y^2 - x^2 = x^3 + y^3 \qquad \text{......(1)}$$

Putting $y = mx$, we get

$m^2x^2 - x^2 = x^3 + m^3x^3$ or $m^2 - 1 = x + m^3x$

Now as $x \to 0$, $m^2 - 1 = 0$ or $m = \pm 1$.

$\therefore$ the equations of the tangents are

$$y = x \text{ and } y = -x.$$

Thus we find that *the linear factors of the terms of the lowest degree equated to zero give the equations of the tangents at the origin.*

Note. Differentiating equation (i) with respect to x, we have

$$2y\frac{dy}{dx}-2x=3x^2+3y^2\frac{dy}{dx}, \quad \therefore \frac{dy}{dx}=\frac{2x+3x^2}{2y-3y^2}$$

which assumes the meaningless form $\frac{0}{0}$ for the origin $x=0$, $y=0$. The present method is helpful in this case.

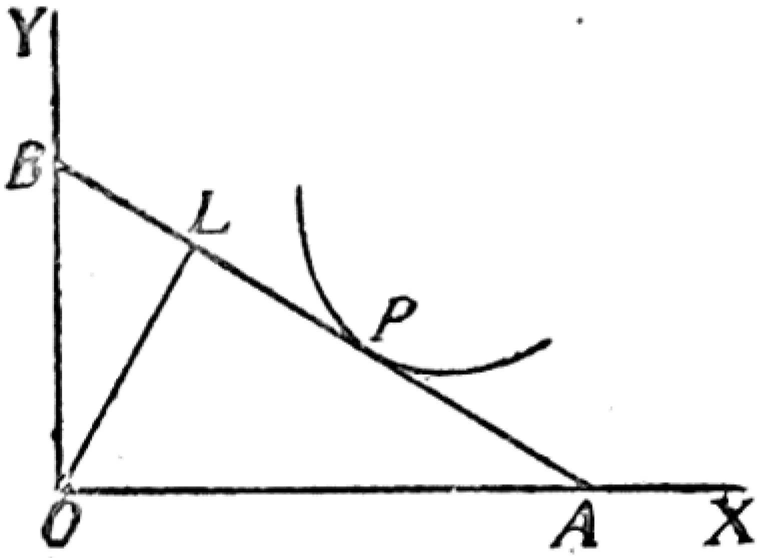
5·44. Intercept of a tangent on the co-ordinate axes are obtained by solving the equation of the tangent.

$$y-y'=\frac{dy'}{dx'}(x-x') \qquad \text{...(1)}$$

with the axes $y=0$ and $x=0$

By putting $y=0$ in (1), we get

$$-y'=\frac{dy'}{dx'}(x-x')$$



This gives $\mathbf{OA}=x=\mathbf{x'-y'\frac{dx'}{dy'}}$ ...(2)

Again by putting $x=0$ in (1), we have

$$\mathbf{OB}=y=\mathbf{y'-x'\frac{dy'}{dx'}} \qquad \text{...(3)}$$

Now since $AB^2=OA^2+OB^2$, we can find AB, *the length of the portion of the tangent intercepted between the co-ordinate axes.*

5·441. The length of the perpendicular from the origin to the tangent to a curve at any point can be written down by the usual method for the perpendicular from a given point to a given straight line.

In this general case the length of the perpendicular from the origin (o, o) to the tangent (1) is given by

$$\text{OL}=\left(x'\frac{dy'}{dx'}-y'\right)\Big/\sqrt{1+\left(\frac{dy'}{dx'}\right)^2} \qquad \text{......(4)}$$

Note. In numerical calculations it will be easier to obtain the results directly from the equation of the tangent and results (2), (3) and (4) need not, therefore, be remembered as formulae

Example. Find the equation of the tangent at any point (x, y) of the parabola.

$$x^{\frac{1}{2}}+y^{\frac{1}{2}}=a^{\frac{1}{2}}.$$

Obtain the length of the portion of the tangent intercepted by the co-ordinate axes and the length of the perpendicular from the origin to the tangent.

Prove also that the sum of the intercepts on the co-ordinate axes is constant,

$$\text{we have, } x^{\frac{1}{2}}+y^{\frac{1}{2}}=a^{\frac{1}{2}}$$

$$\text{and } \tfrac{1}{2}x^{-\frac{1}{2}}+\tfrac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx}=-\frac{x^{-\frac{1}{2}}}{y^{-\frac{1}{2}}}=-\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}.$$

We take (X, Y) to be the co-ordinates of any point on the tangent to distinguish it from (x, y) any point on the curve.

Thus the equation of tangent is

$$Y-y=-\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}(X-x)$$

$$\text{or } x^{\frac{1}{2}}Y+y^{\frac{1}{2}}X=x^{\frac{1}{2}}y+y^{\frac{1}{2}}x=x^{\frac{1}{2}}y^{\frac{1}{2}}(x^{\frac{1}{2}}+y^{\frac{1}{2}})$$

$$=x^{\frac{1}{2}}\,y^{\frac{1}{2}}\,a^{\frac{1}{2}} \qquad \text{[by (1)]} \qquad \ldots(2)$$

Putting Y=0 in (2), we get

$$X=\text{intercept on the } x\text{-axis}=x^{\frac{1}{2}}a^{\frac{1}{2}} \qquad \ldots(3)$$

Putting X=0 in (2), we get

$$Y=\text{intercept on the } y\text{-axis}=y^{\frac{1}{2}}\,a^{\frac{1}{2}} \qquad \ldots(4)$$

$\therefore$ the sum of the intercepts on co-ordinate axes

$$X+Y=a^{\frac{1}{2}}\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)=a^{\frac{1}{2}}\times a^{\frac{1}{2}}=a, \text{ a constant.}$$

Portion of the tangent intercepted between the co-ordinate axes $=\sqrt{X^2+Y^2}=xa+ya=a(x+y)$.

Length of the perpendicular for (0,0) to the tangent

$$(2) \qquad =\frac{x^{\frac{1}{2}}y^{\frac{1}{2}}a^{\frac{1}{2}}}{\sqrt{x+y}}=\sqrt{\frac{axy}{x+y}}.$$

Exercise XIV

1. Find the equations of the tangent and normal at the point t to the curve given by $x=at^2$, $y=2at$. (P. U. 1952)

2. Find the equation of the tangent at the point t on the curve $x=at^2$, $y=at^3$.

3. Find the equation of the tangent to the curve

$$x=t^2, \quad y=t^2-2t, \text{ at } t=1.$$

4. Find the equation of the tangent at the point t to the curve

$$x=1-t, \qquad y=t-t^2.$$

Also find the equations of the tangent at the origin and the tangent parallel to the axis of x.

5. Find the equation of the normal at θ to the curve

$$x=a\cos\theta, \qquad y=b\sin\theta. \qquad \text{(P. U. 1952 S.)}$$

6. Find the equation of the tangent and normal at the point determined by θ on the cycloid

$$x=a(\theta+\sin\theta), \quad y=a(1-\cos\theta) \qquad \text{(P. U. 1949 S.)}$$

7. Show that the equation of the tangent at the point t to the curve $x=\dfrac{t}{1-t^2}$, $y=\dfrac{t^2}{1-t^2}$ is

$$(1+t^2)y-2tx+t^2=0.$$

D **8.** Find the lengths of the perpendiculars from (0,0) and (6,0) to the tangent at any point of

$$x=t^2, \; y=2t.$$

9. Write down the equations of the tangents at the origin to the curves :

(*i*) $y=2x+3x^2+4x^3$ (*ii*) $x^2=3y+x^4$
(*iii*) $y^2=x(1+x^2)$ (*iv*) $(x-y)^2=x-3y$
(*v*) $y^2=x^2(1+x)$ (*vi*) $x^3+y^3=3axy$
(*vii*) $y^2=4x^2+x^3$ (*viii*) $x^3=y(x+y)$.

10. Why is it impossible to write down the equation of the tangent at the origin to

(*i*) $y=3+4x+x^2$ (*ii*) $x^2+y^2=x^3+y^3$?

11. Show that the sum of the intercepts of the tangent at any point to the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ on the axes is constant. (P. U. 1952)

12. Find the length of the perpendicular from the origin on the tangent at any point (x, y) of the curve $x^4+y^4=c^4$. (P. U.)

13. Find the abscissa of the point on the curve $ay^2=x^3$, the normal at which cuts equal intercepts from the co-ordinate axes. (P. U. 1951)

14. Find the equation of the tangent to curve

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$$

and show that the portion of the tangent intercepted between the axes of co-ordinates is constant.

(P. U. 1950)

Also find the length of the perpendicular from the origin on the tangent and the area of the triangle formed by the tangent and the co-ordinate axes.

15. Find the equations of the tangent and normal at the point t on the curve $x=a\cos^3 t$, $y=a\sin^3 t$.

Prove that the portion of the tangent intercepted between the axes is of constant length a.

(P. U. 1948)

[Note that the curves of Q. 14 and 15 are the same.]

5·5. Angle of intersection of two curves :—*When two curves intersect, the angle between them at a common point is the same as the angle between the tangents drawn to the two curves at that point.*

If m, m' be the slopes of the tangents to the two curves at a point of intersection and if θ be the angle, between them, then

$$\tan\theta=\frac{m-m'}{1+mm'}.$$

the angle θ being acute or obtuse according as the right-hand side is positive or negative.

The acute angle between the tangents is taken to be the angle of intersection of the two curves.

When $mm'=-1$, $\theta=90^\circ$ and the curves cut each other at right angles and are said to be *orthogonal.*

Examples

Ex. 1. Find the angle between the curves $x^2+y^2=5$ and $y^2=4x$ at their point of intersection in the first quadrant.

The abscissae of the points of intersection of the given curves are given by

$$x^2+4x=5 \text{ or } (x+5)(x-1)=0 \quad \therefore \text{ i.e. } x=1 \text{ or } -5,$$

$\therefore$ their point of intersection in the first quadrant is $(1, 2)$

Differentiating the given equations, we have

$$2x+2y\frac{dy}{dx}=0 \quad \therefore\ m=\frac{dy}{dx}=-\frac{x}{y}=-\tfrac{1}{2} \text{ at } (1,2)$$

$$2y\frac{dy}{dx}=4 \quad \therefore\ m'=\frac{dy}{dx}=\frac{2}{y}=1 \text{ at } (1,2)$$

Hence θ, the angle of intersection at $(1, 2)$ is given by

$$\tan\theta=\frac{m'-m}{1+mm'}=\frac{1+\frac{1}{2}}{1-\frac{1}{2}}=3$$

$$\therefore\ \theta=\tan^{-1}3=71^\circ\ 34'.$$

Ex. 2. Find the condition that the curves

$$ax^2+by^2=1 \text{ and } a'x^2+b'y^2=1$$

may cut orthogonally (P. U.)

Let (x', y') be a point of intersection of the two curves.

Differentiating the first equation, we have

$$2ax+2by\frac{dy}{dx}=0 \text{ i.e., } \frac{dy}{dx}=-\frac{ax}{by}.$$

So that m, the slope of the first curve

at the point $(x', y')=-\dfrac{ax'}{by'}$

Similarly m', the slope of the second curve at the

same point $=-\dfrac{a'x'}{b'y'}$

The two curves are orthogonal if

$$-1=mm'=\left(-\frac{ax'}{by'}\right)\times\left(-\frac{a'x'}{b'y'}\right)=\frac{aa'}{bb'}\cdot\frac{x'^2}{y'^2} \qquad ..(1)$$

But (x', y') lies on both the curves,

$\therefore\ ax'^2+by'^2=1$ and $a'x'^2+b'y'^2=1$

By substraction, we get

$$(a-a')x'^2+(b-b')y'^2=0, \text{ i.e., } \frac{x'^2}{y'^2}=-\frac{b-b'}{a-a'}.$$

$\therefore$ From (1), we get

$$-1=-\frac{aa'(b-b')}{bb'(a-a')}, \text{ i.e. } \frac{a-a'}{aa'}=\frac{b-b'}{bb'}$$

which gives $\dfrac{1}{a'}-\dfrac{1}{a}=\dfrac{1}{b'}-\dfrac{1}{b}$ or $\dfrac{1}{a}-\dfrac{1}{b}=\dfrac{1}{a'}-\dfrac{1}{b'}$.

Exercise XV

1. Find the angles of intersection of each of the following pairs of curves.

(*i*) $x^2+y^2-2x=0,\ y^2=4x.$

(*ii*) $y^2=x,\ x^2=y$ (P. U. 1949)

(*iii*) $y=x^3,\ y=\dfrac{1}{x}$

(*iv*) $y=x^3,\ 6y=7-x^2$ (P. U. 1952)

(*v*) $y=\sin 2x,\ y=\sin x.$

(*vi*) $y^2=2ax,\ y^2=a^2-x^2$ (D. U.)

2. Show the following pairs of curves cut each other orthagonally :—

(*i*) $x^2-y^2=a^2$, $xy=c^2$. (*ii*) $x^2-y^2=5$, $4x^2+9y^2=72$
(*iii*) $x^3-3xy^2=a^3$, $3x^2y-y^3=b^3$

3. Find the angle of intersection of the curves.

$x^2-y^2=a^2$ and $x^2+y^2=a^2\sqrt{2}$ (P. U.)

4. Find the angle at which the curve $y=\dfrac{x}{1+x}$ cuts the axis of x. (P. U.)

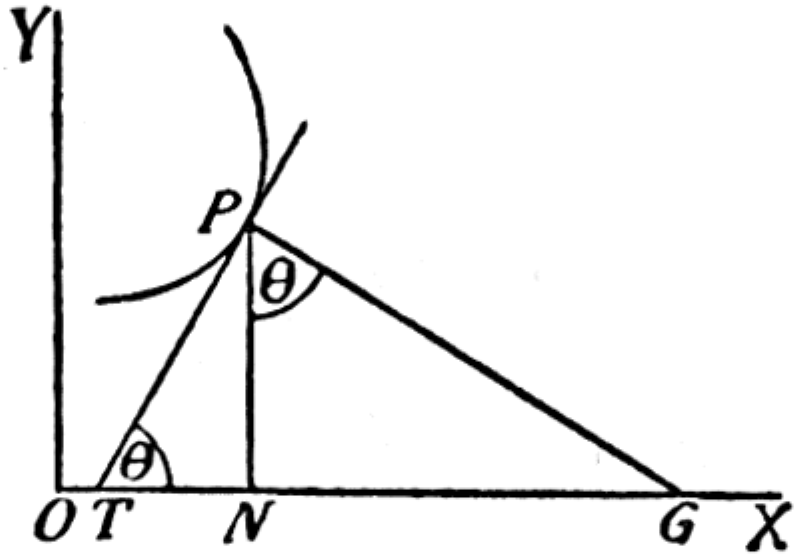
5. Prove that the tangents to the curves, $y^2=4x^3$ and $x^2+y^2=5$ at their point of intersection lying in the first quadrant are inclined to one another at an angle $\tan^{-1} 7$.

6. Find the condition that the conics

$$lx^2+my^2=1 \quad \text{and} \quad l'x^2+m'y^2=1$$

may cut orthogonally. (P. U. 1952)

5·6. Length of the tangent normal, sub-tangent, and subnormal.



Let P (x, y) be any point on the curve $y=f(x)$. The tangent and normal at P intersect the X-axis in T and G and NP$=y$ is the ordinate of the point P.

Let the tangent PT make with the X-axis an angle θ, then $\angle$NGP$=\angle$PTN$=\theta$.

and $$\tan\theta=\frac{dy}{dx}.$$

From the triangles PNT and PNG, we have

(*i*) **Length of the tangent**$=$PT$=y \operatorname{cosec}\theta$.

$$=y\sqrt{1+\cot^2\theta}=\frac{y\sqrt{1+\tan^2\theta}}{\tan\theta}=\frac{y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}{\dfrac{dy}{dx}}$$

(*ii*) **Length of the normal**$=PG=y \sec. \theta$.

$$=y\sqrt{1+\tan^2\theta}=y\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

(*iii*) **Length of the sub-tangent**$=NT=y \cot \theta$.

$$=\frac{y}{\tan\theta}=\frac{y}{\frac{dy}{dx}}.$$

(*iv*) **Length of the sub-normal**$=NG=y \tan\theta=y\frac{dy}{dx}$.

Examples

Ex. 1. Find the lengths of the sub-normal, sub-tangent, normal and tangent of the curve $y=5x^3-2x+1$ at the point where the curve cuts the axis of y.

Putting $x=0$ in the equation of the curve, we get $y=1$, so that the curve cuts the axis of y at the point $(0, 1)$.

$$\text{Now } m=\frac{dy}{dx}=15x^2-2=-2 \text{ at } (0, 1).$$

$$\therefore \text{ the length of the sub-normal}=y\frac{dy}{dx}=y\times m$$

$$=1\times(-2)=2 \text{ in magnitude}$$

The length of the subtangent$=y/m=-\frac{1}{2}=\frac{1}{2}$ in magn.

The length of the normal$=y\sqrt{1+m^2}=1\sqrt{1+4}=\sqrt{5}$

$$\text{The length of the tangent}=\frac{y\sqrt{1+m^2}}{m}=\frac{1\sqrt{1+4}}{-2}$$

$$=\frac{\sqrt{5}}{2} \text{ in magn.}$$

Ex. 2. *Show that the sub-normal at any point of the parabola* $y^2=4ax$ *is of constant length.* (P. U. 1950)

We have, $y^2=4ax$ and $2y\frac{dy}{dx}=4a$ *i. e.* $\frac{dy}{dx}=\frac{2a}{y}$

$\therefore$ the length of the sub-normal at any point (x, y) $=y\frac{dy}{dx}=y\times\frac{2a}{y}=2a$, a constant equal to the length of the semi-latus rectum of the parabola.

Ex. 3. Find the lengths of the sub-tangent, sub-normal, tangent, normal and perpendicular from the origin on the tangent at the point (b, b) for the curve

$$y^2=\frac{x^3}{2b-x}.$$

Differentiating the equation of the curve with respect to x, we have

$$2y\frac{dy}{dx}=\frac{3x^2(2b-x)-x^3(-1)}{(2b-x)^2}=\frac{6bx^2-2x^3}{(2b-x)^2}$$

$$\therefore \qquad \frac{dy}{dx}=\frac{x^2(3b-x)}{y(2b-x)}=2 \text{ at the point } (b, b)$$

Length of the sub-tangent$=y\div\dfrac{dy}{dx}=\dfrac{b}{2}$

Length of the sub-normal$=y\times\dfrac{dy}{dx}=2b$

Length of the tangent $\quad =y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}\div\dfrac{dy}{dx}$

$$=b\sqrt{1+4}\div 2=\frac{\sqrt{5}}{2}b.$$

Length of the normal$=y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}=b\sqrt{1+4}$

$$=\sqrt{5}b.$$

The equation of the tangent at (b, b) is

$$y-b=\frac{dy}{dx}(x-b)=2(x-b)$$

i. e., $\qquad y-2x+b=0$

$\therefore$ The length of the perpendicular from the origin

to this tangent$=\dfrac{0-0+b}{\sqrt{1+4}}=\dfrac{b}{\sqrt{5}}$

Exercise XVI

1. Find the length of the sub-tangent and sub-normal at the point (1, 9) of the curve $y=4x^2+2x+3$.

2. Find the length of the tangent and normal at the point $(1, \frac{1}{2})$ on $4y=x+x^2$, intercepted between the point and the axis of x. (D. U.)

3. Find the lengths of the tangent, sub-tangent, normal and sub-normal for the curve $y=3x+x^2$ at the point (1,4).

4. Find the points on the curve $y=x^2-5x+6$ at which the sub-tangent and the sub-normal are of equal length.

5. Find the lengths of the sub-tangent and sub-normal at $\phi=\frac{1}{4}\pi$ to the curve $x=a \cos \phi$, $y=b \sin \phi$. (P. U. 1944)

6. Find the lengths of the tangent and normal at the point θ of the curve $x=a \cos^3\theta$, $y=a \sin^3\theta$.

7. Find the lengths of the sub-tangent and sub-normal at the point t on the cycloid.

$$x=a\,(t+\sin t), \qquad y=a(1-\cos t)$$

8. What should be the value of n in the equation to the curve $a^{n-1}\, y=x^n$ in order that the sub-normal may be of constant length.

9. In the curve $x^{m+n}=a^{m-n}.\ y^{2n}$, prove that the mth power of the sub-tangent varies as the nth power of the sub-normal.

10. Through what points of the ellipse $\frac{x^2}{16}+\frac{y^3}{9}=1$ must a tangent and a normal be drawn in order that they may form with the axis of x as base an isosceles triangle ? (*D.U.*)

Also prove that the length of the normal varies inversely as the perpendicular from the origin on the tangent.

11. Show that in the case of the curve $\beta y^2=(x+\alpha)^3$, the square of the sub-tangent varies as the sub-normal. (Aligarh)

12. Show that for a parabola, the ratio of the sub-tangent to the sub-normal varies as the abscissa.

13. Show that for any *cartesian curve*, the ratio of the normal to the tangent is equal to the gradient of the curve.

14. Prove that in any curve

$$\frac{\text{Sub-normal}}{\text{Sub-tangent}}=\left(\frac{\text{Normal}}{\text{Tangent}}\right)^2.$$

15. Prove that the perpendicular drawn from the foot of the ordinate to the tangent of a curve is $y/\sqrt{1+(dy/dx)^2}$ (P.U. 1950)

Revision Questions II

1. Obtain the equation of the tangent at a point to the curves.

(1) $y=f(x)$; (2) $x=f(t)$; $y=\phi(t)$. (P.U. 1946)

2. Obtain the equation of the tangent to the curve $y=5x^2-2x+3$ at the point $(1,6)$ and determine the angle at which the curve cuts the y-axis (P.U)

3. Show that the equation of tangent to the curve $3ay^2=x^2(x+a)$ at $(2a, 2a)$ is $3y=4x-2a$.

Determine the point where it meets the curve again and show that it is the normal at that point. (P.U. 1940)

4. Show that the part of the tangent to the hyperbola $xy=a^2$ included between the co-ordinate axes is bisected at the point of contact.

5. Find the values of a, b and c so that the curve $y=ax^2+bx+c$ may pass through the point $(-1, 4)$ and may touch the line $x+2y=4$ at $(-2,3)$

6. Two curves $xy=$constant and $yx^2=$constant pass through the point $P(3,4)$. Find the angle at which they intersect.

Find also the length intercepted on the axis of x between the two normals drawn to the curves at P. (P.U.)

7. Find the equation of the tangent to the curve $x=3t-10$, $y=7+t^3$ at the point where $t=1$; show that this tangent cuts the curve again at the point where $t=-2$

8. Show that the curves $y^3=x$ and $y^3=x^3$ intersect at an angle $\tan^{-1}\frac{4}{7}$ at their point of intersection other than the origin.

9. Show that the tangent at the point (2,1) on the curve $x^3+y^3=9$ cuts the same curve again at the point $(\frac{20}{7}, -\frac{11}{7})$. What angle does this tangent make with the curve at the second point ?

10. Show that the normal to the curve $y^2=4x$ and the tangent to the curve $4x^2=y^3$ at the point of intersection of the curves are inclined to one another at an angle $\tan^{-1} 8$.

11. Show that the angle of intersection of the curves $y^2=4ax$ and $ay^2=4x^3$ is $\tan^{-1}\frac{1}{2}$.

Also if PG_1, PG_2 be the normals to the two curves at a common point of intersection (other than the origin), meeting the x-axis in G_1 and G_2, then $G_1G_2=4a$.

12 Find the equation of the tangent at (2,2) to the curve $y=(x^2+2)/(x^2-1)$ and determine the point at which it intersects the curve. (P.U.)

13. Find the equations of the tangent and the normal to the allipse $9x^2+25y^2=225$ at the point $\left(\frac{5}{2}, \frac{3\sqrt{3}}{2}\right)$ (P.U. 1953)

14. Prove that $\frac{x}{a}+\frac{y}{b}=2$ touches the curve $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2$ at the point (a, b) for all values of n.

15. Find the normal to the curve $\sqrt{xy}=a+x$, which makes equal intercepts on the co-ordinate axes. (P. U. 1942)

CHAPTER VI

DERIVATIVE AS RATE OF CHANGE

"All things are in a state of change nothing remains unaltered for an instant. It is a moving, changing universe with which we have to do."

6·1. The idea of *rate* of change occurs constantly in everyday experience. We have seen that changes in one variable are related to the changes in other varibles. The Differential calculus is the study of relationship between the rates of change of related variables.

6·11. Application of derivative to rectilinear motion. When a body moves along a straight line, the distance travelled changes with the time. Whenever a relation between s the distance travelled and t the time taken is known, the distance travelled during any time can be obtained by the substitution of the proper value of t; for example if $s=16t^2$, $s=256$ ft. when $t=4$ sec. and $s=144$ ft. when $t=3$ sec., so that the distance travelled in the 4th second is $256-144=112$ ft.

6·2. Uniform and Variable Velocity.

The distance travelled is not the only thing which we wish to enquire about a body in motion. The other important things about it are :—

(*i*) The rate of change of distance, *i.e.*, its speed at any time, and

(*ii*) Whether that speed is uniform or variable and if variable, the rate of change of velocity, or *acceleration*, of the body.

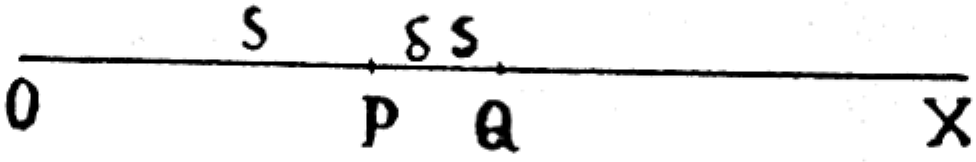
If a railway train, for example, in going from one station to another, travels a distance of 80 miles in four hours, it is said to be moving with a speed of 20 miles per hour on the average. Had the train been moving uniformly throughout its motion, 20 miles per hour would have been its velocity at any time during its motion.

But we know that before it left the first station, it was stationary and then its velocity increased gradually, *i.e.*, the train moved with some *acceleration* in the beginning. After that we may suppose the train to have moved with a constant velocity for some time and then, before stopping at the next station, it must have decreased its velocity, in other words, it must have moved with some *retardation*.

Under these circumstances we may say that twenty miles an hour is the average velocity of the train and about its actual velocity at any particular instant nothing can be said definitely.

We now proceed to show how with the aid of Calculus a definite meaning can be gvien to the velocity at a particular time t, when the velocity is variable.

6.21. Measure of the variable velocity



Suppose a body begins to move from the point O along the straight line OX and let it come in the position P after describing a distance s in time t.

Let it further move a little distance δs, in a little time δt, and come in the position Q. $\frac{\delta s}{\delta t}$ is then the average velocity of the body during the interval δt. As δt and consequently δs becomes smaller and smaller, the point Q comes nearer and nearer to the point P and $\frac{\delta s}{\delta t}$ must more and more approximate to the value of the velocity at point P.

Hence the velocity at P at the instant $t = \underset{\delta t \to 0}{\mathrm{Lt}} \frac{\delta s}{\delta t}$

$$= \frac{ds}{dt}$$

Thus *the derivative of s with respect to t is the measure of the velocity at the time t.*

Note (*i*) $\frac{ds}{dt}$, the velocity at any time, is the rate of change of the distance s with respect to time t. Similarly $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are rates of change of x and y with respect to t.

The phrase "*rate of change*" is used most often in reference to *time*, and the words "with respect to t", are often omitted.

(*ii*) We know that $\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}$,

i.e., rate of change of $y=\frac{dy}{dx}\times$(rate of change of x).

(*iii*) As $\frac{dy}{dt}$ is the rate of change of y with respect to t, similarly $\frac{dy}{dx}$ may be regarded as the rate of change of y with respect to x, and it is possible to consider the rate of change of y relative to x without any reference to time.

6·3. Measure of Acceleration.

We have seen that the rate of change of distance at a particular instant is the velocity at that instant. Similarly the rate of change of velocity (when velocity is not uniform) with respect to time is called the *acceleration* which is generally denoted by f.

Let v be the velocity at any instant t, of a body moving with variable velocity, and δv, the change of velocity in a small interval of time δt, then $\frac{\delta v}{\delta t}$ is the average rate of change of velocity in time δt.

Making δt very small, we get rate of change of velocity at the instant t, *i.e.*, the acceleration at the instant t

$$=\underset{\delta t\to 0}{\text{Lt}}\frac{\delta v}{\delta t}=\frac{dv}{dt}.$$

$$\text{But } v=\frac{ds}{dt}. \quad \therefore f=\frac{dv}{dt}=\frac{d}{dt}\left(\frac{ds}{dt}\right)=\frac{d^2s}{dt^2}.$$

Thus the *acceleration of the body is either the first derivative of the velocity or the second derivative of the distance with respect to time.*

Examples

Ex. 1. The distance-time formula of a body moving in a straight line is

$$s=\tfrac{1}{6}t^3-3t^2-14t \quad \text{...............(1)}$$

(s being in feet and t in seconds) and the motion starts when $t=0$. Find the velocity after time t and the time after which the body is instantaneously at rest.

Find also the velocity and acceleration after 8 sec.

The velocity v at any time t is given by

$$v=\frac{ds}{dt}=\tfrac{1}{6}(3t^2)-3(2t)-14=\tfrac{1}{2}t^3-6t-14$$

when $t=8$, $v=-30$ ft. per sec.

The time when the body is at rest is given by

$$0=\tfrac{1}{2}t^2-6t-14 \text{ or } (t-14)(t+2)=0$$

which gives $t=14$ sec.,

The negative value of t, given by $t+2=0$ is rejected.

Now the acceleration f at any time t is given by

$$f=\frac{dv}{d} \text{ or } \frac{d^2s}{dt^2}=\tfrac{1}{2}(2t)-6=t-6$$

When $t=8$, $f=8-6=2$ ft. per sec. per sec.

Ex. 2. The radius x of a cylinder is increasing at the rate of 2 ft. per sec. and its altitude y is decreasing at the rate of 3 ft. per sec. Find the rate of change of its volume when $x=3$ ft. and $y=5$ ft.

Let V cu. ft. be the volume of the cylinder.

Then $$V=\pi x^2y,$$

and the rate of change of the volume is given by,

$$\frac{dV}{dt}=\pi x^2\frac{dy}{dt}+\pi y\cdot 2x\frac{dx}{dt}$$

By hypothesis, $\frac{dx}{dt}=2$ and $\frac{dy}{dt}=-3$

The sign is negative in the second case as the altitude y is given to be decreasing.

Hence $\frac{dV}{dt}=\pi x^2(-3)+2\pi xy(2)=4\pi xy-3\pi x^2$

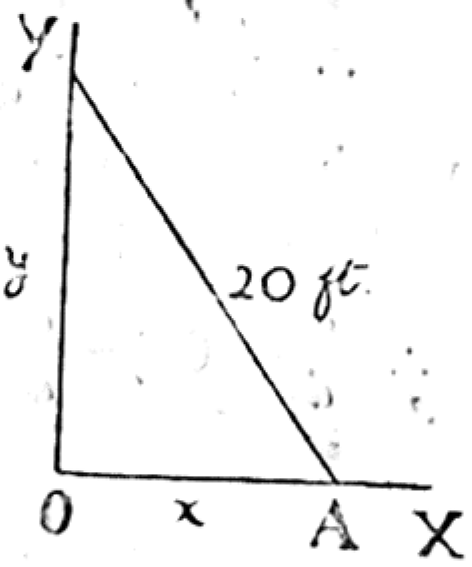
This gives the rate of increase of the volume in general terms.

In the particular case when $x=3$, $y=5$, we have

$$\frac{dV}{dt}=4\pi 15-3\pi\cdot 9=\pi(60-27)=33\pi$$

Ex. 3 A ladder 20ft. long leans against a wall while its base is drawn away from the wall at the rate of 2ft./sec. How fast is the top of the ladder descending when its height above the floor is 16 ft.

Let x be the distance of the lower end of the ladder from the wall and let y be the height of the upper end from the floor, then



$$x^2+y^2=(20)^2=400 \qquad \dots\dots\dots (1)$$

Differentiating with respect to the time t, we have

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0 \quad i.\ e.\ \frac{dy}{dt}=-\frac{x}{y}\frac{dx}{dt} \quad (2)$$

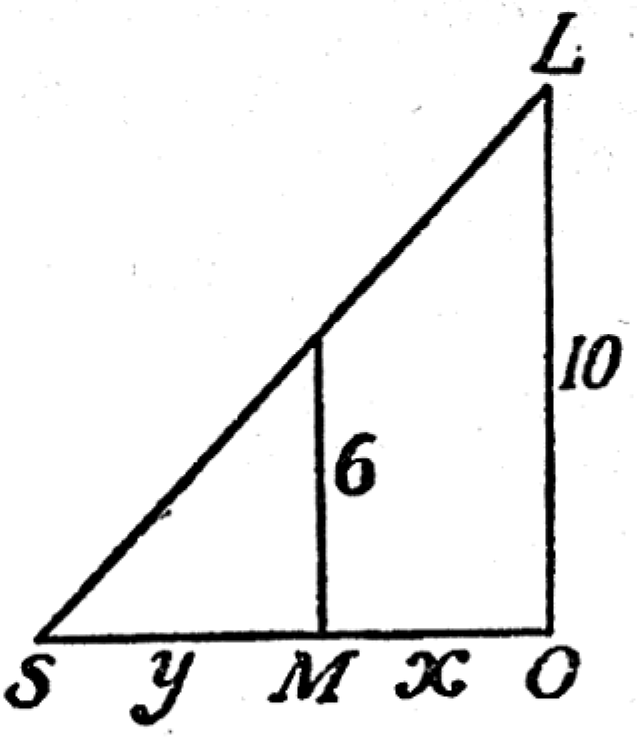
Now when $y=16$, $x=12$ from equation (1).

$\therefore$ Substituting $y=16$, $x=12$ and $\frac{dx}{dt}=2$ ft./sec.

in (2), we get $\frac{dy}{dt}=-\frac{12}{16}(2)=-\frac{3}{2}$ft./sec.

The sign being negative, it indicates that the upper end descends at the rate of $\frac{3}{2}$ft. per sec.

Ex. 4. A man 6 ft. tall is walking directly away from a street lamp 10 ft. above the ground, at the rate of 176 ft. per minute. Find the rate at which his shadow lengthens and the velocity of the end of the shadow.



Let y be the length of the man's shadow and x, his distance from lamppost.

$$\text{Then} \quad \frac{y}{6}=\frac{x+y}{10}$$

$$\text{or } 4y=6x \quad i.\ e.\ \ y=\tfrac{3}{2}x$$

$$\therefore \quad \frac{dy}{dt}=\frac{3}{2}\frac{dx}{dt}=\tfrac{3}{2}\times 176=264 \text{ ft. per min.}$$

which gives the rate of lengthening of the shadow

The velocity of the end of the shadow

$$\therefore \quad \frac{d}{dt}\,(\text{OS})=\frac{d}{dt}\,(x+y)=\frac{dx}{dt}+\frac{dy}{dt}=176+264$$

$$=440 \text{ ft./min.}$$

Exercise XVII

1. A particle moves in a straight line in such a way that its distance from a fixed point O in the line at the end of t seconds is $11+5t+t^3$ feet. Find (i) the distance from O (ii) its speed and (iii) its acceleration at the end of 4 seconds.

2. If the distance time formula for a body moving along a straight line is $s=t^2-2t$, show that the body is travelling four times as fast when $t=5$ as when $t=0$ and in the opposite direction.

3. A ball is thrown vertically upwards and after t sec. its height h ft above the ground is given by

$$h=80t-16t^2$$

(a) Show that it reaches the ground again after 5 sec.

(*b*) Find its velocity after (*i*) 1 sec., (*ii*) 3 sec. and explain the signs of these velocities.

(*c*) Find the velocity with which it was thrown up and the time after which the ball is instantaneously at rest.

4. The side of a square increases uniformly at the rate of 1 inch per sec. At what rate is the area increasing when the side is 6 inches ?

5. At what point on $y^2=x^3$ does y increase with x at the same rate as x increases with y ?

6. A rod AB, 10 ft. long moves with its ends A and B on two perpendicular lines OX and OY respectively. If A is 8 ft. from O and is moving away at the rate of 2 ft. per sec. Find at what rate the end B is moving. (Madras)

7. A pencil 6 inches long is held horizontally with its centre x ft, below an electric lamp. Find the length y ft. of the shadow cast by the pencil on a table 8 ft. below the light.

Find the rate of change of the length of the shadow when the pencil is 4 ft. below the lamp and is (*a*) lowered (*b*) raised at 2 ft. per sec.

8. Show that the volume of a spherical soap bubble increases $4\pi r^2$ times as fast as the radius r.

Find the rate of increase of the (*a*) volume and (*b*) the surface area when $r=2\frac{1}{2}$ inches and is increasing at the rate of $\frac{1}{3}$ inches per sec.

9. Find the angle x, which increases twice as fast as its sine. (P. U. 1949)

10 The volume of a cube is increasing at a constant rate. Prove that the surface increase varies inversely as the length of the edge. (D. U.)

11. A stone dropped into a calm lake causes a series of circular ripples. The radius of the outer one increases at 2 ft. per sec. How rapidly is the disturbed area changing at the end of 3 sec?

12. Water is running from a cylindrical tank 3 ft. in diameter at the rate of $3\pi\sqrt{h}$ cu. ft. per minute; when h is the depth in feet of the water in the tank. How fast is the surface of the water falling when h is equal to 9 ft. ?

13. The top of a ladder 20 ft. long slides down a vertical wall. Find the ratio of the speeds of the top and bottom when the ladder makes an angle of 60° with the ground.

14. Two men A and B start from a place O to walk in perpendicular directions ; A at 3 miles per hour ond B at $4\frac{1}{2}$ miles per hour.

At what rate are they receding from each other after two hours ?

15. A ladder AB, 26 ft. long, rests with one end A on the ground OA and with the other projecting over a wall OC, 12 feet high. The end A is pushed along the ground towards the wall at 3 ft. per sec. Find the vertical velocity of B when A is 5 ft. from the wall.

6·3 Application of differential co-efficient in approximate calculations.

It is often required to calculate the effect upon one quantity of a small change in a related quantity.

If a known formula connects the quantities x and y, and if δy is the change in the value of y corresponding to a change δx in the value of x, we can calculate δy in terms of x and δx.

Since $\dfrac{dy}{dx}$ is the limiting value of $\dfrac{\delta y}{\delta x}$ when $\delta x \to 0$,

we have
$$\frac{\delta y}{\delta x}=\frac{dy}{dx}+\alpha$$

where α tends to zero as δx tends to zero

$$\therefore\ \delta y=\frac{dy}{dx}\,\delta x+\alpha\,\delta x$$

Now when δx is made sufficiently small, $\alpha\delta x$, being a small quantity of the second order may be neglected and we may write

$$\delta y=\frac{dy}{dx}\,\delta x \text{ (approximately)} \qquad \text{......(1)}$$

If $y=f(x)$, then $\frac{dy}{dx}=f'(x)$, and equation (1) may also be written as

$$\delta y=f'(x)\ \delta x \qquad \text{......(2)}$$

Thus if δx is the small error in the measured value of x, the error in the function $f(x)$ is $f'(x)\delta x$ approximately.

Examples.

Ex. 1. The valume V of a cube of side x is given by

$$V=x^3. \qquad \text{......(1)}$$

If δx is a small error in the side x, then δV, the error in the volume is given by

$$\delta V=\frac{dV}{dx}\cdot\delta x=3x^2\ \delta x. \qquad \text{......(2)}$$

Suppose the side is 2 inches and the error δx in x is ·01 then $\delta V=3(2)^2(\cdot 01)=\cdot 12$, so that due to an error of ·01 in the side, the volume is approximately 8·12 cu. in. instead of 8 cu. in.

It is interesting to compare the approximate value obtained above with the exact value obtained as follows ;—

$$V=x^3,\quad V+\delta V=(x+\delta x)^3=x^3+3x^2\ \delta x+3x(\delta x)^2+(\delta x)^3$$

Substituting $x=2$ and $\delta x=\cdot 01$, we get

$$V+\delta V=8+12\times\cdot 01+6\times\cdot 0001+\cdot 000001=8\cdot 120601$$

Thus upto three places of decimals, the approximate value given by equation (2) is the same as the actual value obtained by a longer process.

Ex. 2. Find the error in the area of a triangle calculated by the formula $\frac{1}{2}bc$ sin A due to small error in the value of the angle A.

Given $b=1{\cdot}4$ ft., $c=1{\cdot}8$ft. and $A=60°$, find the error in the area due to an error of 1° in A.

Let y denote the area of the triangle

then $$y=\tfrac{1}{2}bc.\sin A$$

$\therefore$ the error in the area $\delta y=\frac{1}{2}bc\cos A\ \delta A$

Now $b=1.4$ ft., $c=1{\cdot}8$ ft., $A=60°$ and $\delta A=1°$

$$=\frac{\pi}{180}\text{ radian}$$

$$\therefore\ \delta y=\tfrac{1}{2}\times1{\cdot}4\times1{\cdot}8\times\tfrac{1}{2}\times\frac{\pi}{180}=\frac{14\times18}{100\times4\times180}\times\frac{22}{7}$$

$$=\frac{11}{1000}=\cdot011\text{ sq. ft.}$$

Ex. 3. A cylindrical tin can is to be made so as to have its height (x) equal to its diameter.

Find the error in its volume, V, due to a small error in its linear diamensions.

Calculate approximately the relative error in V caused by a percentage error of 0·5 per cent in the diameter and the height.

The height being x, radius of the can$=\frac{1}{2}x$

$$\therefore\quad V=\pi(\tfrac{1}{2}x)^2\,x=\tfrac{1}{4}\,\pi x^3\quad\ldots\quad\ldots\quad\ldots\quad(1)$$

$$\therefore\quad \delta V=\tfrac{1}{4}.3\pi x^2\,\delta x=\tfrac{3}{4}\,\pi x^2\delta x\quad\ldots\quad\ldots\quad\ldots\quad(2)$$

Now δx, the error in $x=\dfrac{0{\cdot}5x}{100}=\dfrac{x}{200}$

$$\therefore\quad\text{the relative error in }V=\frac{\frac{3}{4}\pi\,x^2\delta x}{\frac{1}{4}\pi\,x^3}=\frac{3\delta x}{x}$$

$$=\frac{3}{x}\times\frac{x}{200}=\cdot015$$

Exercise XVIII

1. Each side of a square plate is increased by heat from 3 inches to 3·1 inches; find an approximate value of the increase in the area.

Find also the difference between the actual and the approximate increase.

2. Find an approximate value for $(4{\cdot}005)^3$

3. If θ be changed to $\theta+\delta\theta$, what is the approximate change in $\sin\theta$

Calculate this change when $\theta=60°$ and $\delta\theta=1°$.

4. A circular plate is expanding by heat. When the radius passes through the value 3 ft. it is increasing at the rate of 0·0012 inches. Find the rate of increase of area at this moment. (D. U.)

5. The radius of a sphere is taken as 5 cm. Find the error in its volume and its percentage error if an error of 0·5 mm. is made in measuring the radius.

6. The volume of a gas is measured; the pressure is then computed by the formula $pv=K$. If the allowable error in p is 0·001 K and v can be measured to 0·1 cu. ft., what is the size of smallest container to which the process can be applied.

7. The angle of elevation of the top of the Qutab Minar from a point 120 ft. from its base is found to be 60°. Find the height of the Minar and the percentage error in the calculated height due to an error of 15′ in measuring the angle.

8. The height of a tower is calculated on the assumption that the angle of elevation of its top from a point 200 ft., from the foot is 45°. If an error of 10′ was made in measuring the angle, find the corresponding error made in the height of the tower. Find also the percentage error.

9. A tower on the bank of a river is 120 ft. high, and the angle of elevation of its top from a point on the opposite bank is found to be 30°. If an error of 15′ was made in measuring the angle, find the

corresponding error in the calculated breadth of the river.

10. In a plane triangle if the sides and angles receive small variations, prove that when a and B are constant

$$\tan A.\delta b=-b.\delta A.$$

11. The sides a, b, c and the angle A of a triangle are related by the formula $a^2=b^2+c^2-2bc \cos A$.

(*i*) Find a formula for the percentage error in the side a due to an error of 2 per cent in the value of b, the other terms being supposed correct.

(*ii*) Find the error δa in the side a due to an error δA in the angle A and show that when $b=c=10$ cm. $A=60°$ and $\delta A=1°$, δa is roughly $1\frac{1}{3}$ millimetre.

12. Explain clearly what is meant by the approximate equation $\delta y=f'(x)\,\delta x$, showing how it is derived.

Supposing that the limits of error in measuring the side a of a triangle ABC are $\pm\mu$ per cent where μ is small prove that if the other two sides are correct, the limits of error in calculating the angle A are approximately

$$\pm\frac{1{\cdot}15\mu a^2}{2bc \sin A} \text{ degrees.} \qquad \text{(P. U. 1929)}$$

13. The time (T) of oscillation of a simple pendulum of length l is given by $T^2g=4\pi^2l$. Find the percentage error in T corresponding to an error of 2 percent in the value of l.

14. If $y=f(x)$ and $y+\delta y=f(x+\delta x)$, show that $f'(x)\delta x$ is a good approximation to the value of δy when δx is small.

A step ladder (in the form of the letter A) has equal legs DE, DF, a ft. long, and a string PQ, b ft. in length, is attached to each leg so that $DP=DQ=c$ ft. Find approximately the length δh through which the top D of the ladder is lowered when the string is lengthened by δb ft.

If $a=5$ ft., $b=c=2$ft., $\delta b=1''$; show that $\delta h=\cdot 72''$
(B.U. 1923)

15. The distance of the visible horizon, (x miles) as seen from a height of y feet above sea level is given by the formula $2y=5x^2$.

How many miles can one see from the top of the Qutab Minar which is 250 ft. high ?

Approximately how much higher must one climb in order to see the river Jumna, $\frac{1}{4}$ mile further away ?

CHAPTER VII

INCREASING AND DECREASING FUNCTIONS MAXIMA AND MINIMA

We now proceed to show how calculus may be used to obtain important information concerning functions and their graphs.

7.1. Increasing and Decreasing Functions. *A function $y=f(x)$ is said to be an increasing function of x if, as x increases, the value of the function y or $f(x)$ also increases and it is said to be a decreasing function of x if, while x increases, $f(x)$ decreases.*

In other words $f(x)$ is an increasing function of x, at $x=x_1$,

if $\qquad f(x_2)>f(x_1)$; when x_2 is *just* $>$ x_1,

Suppose a fuction $y=f(x)$ is increasing at x, then for a positive value of δx,

$$f(x+\delta x)>f(x)$$

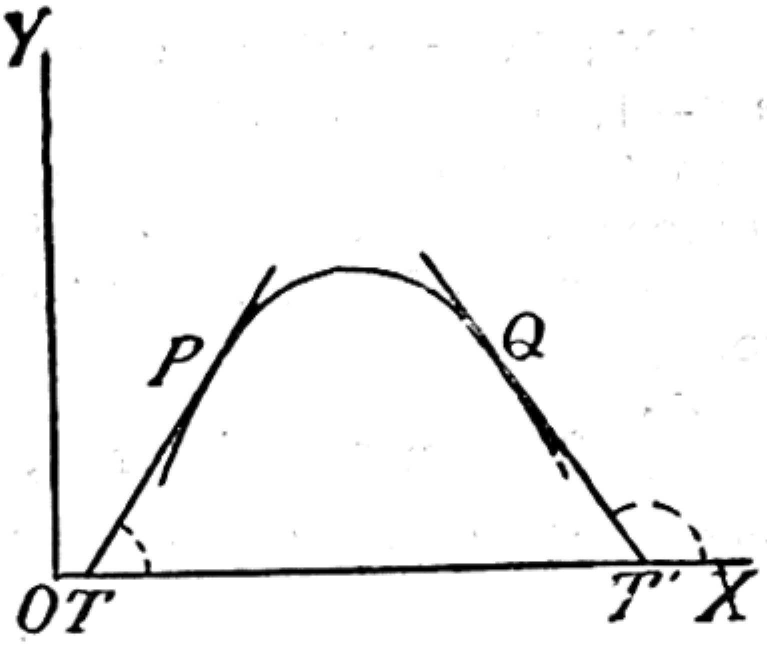
or
$$f(x+\delta x)-f(x)>0$$

or
$$\frac{f(x+\delta x)-f(x)}{\delta x}>0$$

and this is true however small δx may be. Hence proceeding to limit when $\delta x \to 0$, we get

$$\frac{dy}{dx}>0 \quad \text{or} \quad \frac{dy}{dx} \text{ is positive}$$

Similarly if $f(x)$ is decreasing at x, $f(x+\delta x)<f(x)$, when δx is positive and tends to zero, so that $\frac{dy}{dx}$ is negative. Thus *the derivative or $\frac{dy}{dx}$ is positive for an increasing function and is negative for a decreasing function.*



This is obvious geometrically from the graph of a function. In the adjoining figure, at a point such as P where the function is an increasing one, *i.e. increases* as x increases, the tangent makes an acute angle with the axis of x, so that the slope of the curve is positive. Again at a point such as Q where the function is decreasing *i.e. y decreases* as x increases, the tangent makes an obtuse (or negative) angle with the x-axis and therefore the slope is negative.

A point where the slope is neither positive nor negative but is zero, and the tangent is parallel to the x-axis, is known as a **stationary point.**

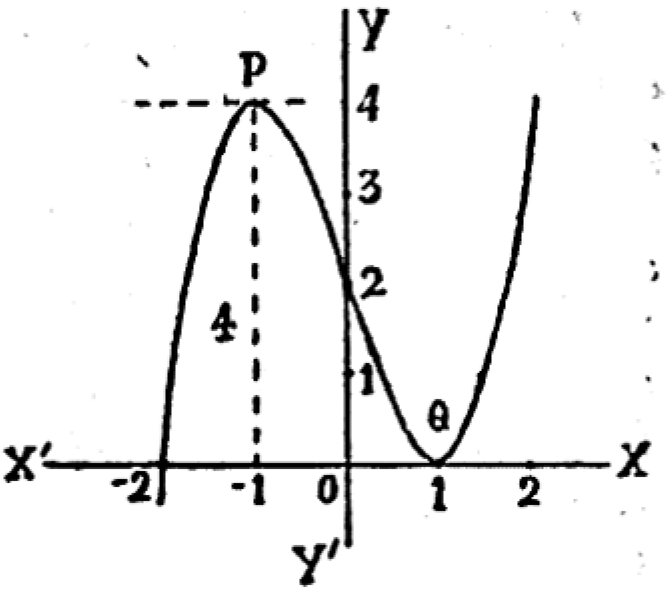
The same function may be an increasing function at one stage and a decreasing function at another stage and may have several such stages, *e.g.*, $y=\cos x$. [See Fig. Ex. 2. Art. 5·2.]

Note. The terms greater and less, increasing or decreasing are used in the algebraic sense, *e.g.*, -7 is greater than -13. The function $y=\frac{1}{x}$ is a decreasing function for all values of x, even for the portion lying in the third quadrant. [See Fig. Ex. 5. Art 5·2]

Example. Show that the curve $y=x^3-3x+2$, increases for all values of x less than -1 and greater than 1 and decreases for values of x from -1 to 1. Show also that it is stationary at the points (1,0) and (−1, 4).

$$\frac{dy}{dx}=3x^2-3=3(x-1)(x+1).$$

This is positive when $x<-1$, and when $x>1$, so that the curve rises and y increases for values of x from $-\infty$ to -1 and from 1 to ∞.



$$\frac{dy}{dx}=0 \text{ when } x=-1 \text{ or } 1,$$

so that the tangents are parallel to the x-axis and the curve is stationary at these points.

When $x=1$, $y=0$, which is the point Q where the axis of x itself is the tangent; and when $x=-1$, $y=4$, which is the point P in the figure.

Exercise XIX.

1. Determine whether y is increasing or decreasing or stationary at the indicated values of x :—

(*i*) $y=3x^2-4$; $x=2$, (*ii*) $y=\cos x$; $x=-\frac{1}{2}\pi, 0, \frac{1}{2}\pi$,
(*iii*) $y=x^3$; $x=-1, 0, 1$, (*iv*) $y=x^4+2x+1$,
$x=-\frac{3}{2}, -1, -\frac{1}{2}, 0$.

2. Show that the curve $y=x^5$ does not decrease and that the curve $y=1/x$ does not increase for any real value of x.

3. Show that the curve $y=x^3+x+7$ has no stationary value and that it rises for all values of x.

4. Show that the graph of $y=x^2+4x$ rises when $x>-2$ and falls when $x<-2$. Indicate this by drawing a rough sketch of the curve.

5. Show that $x^3-9x^2+24x+2$ increases as x increases for values of x less than 2 and greater than 4.

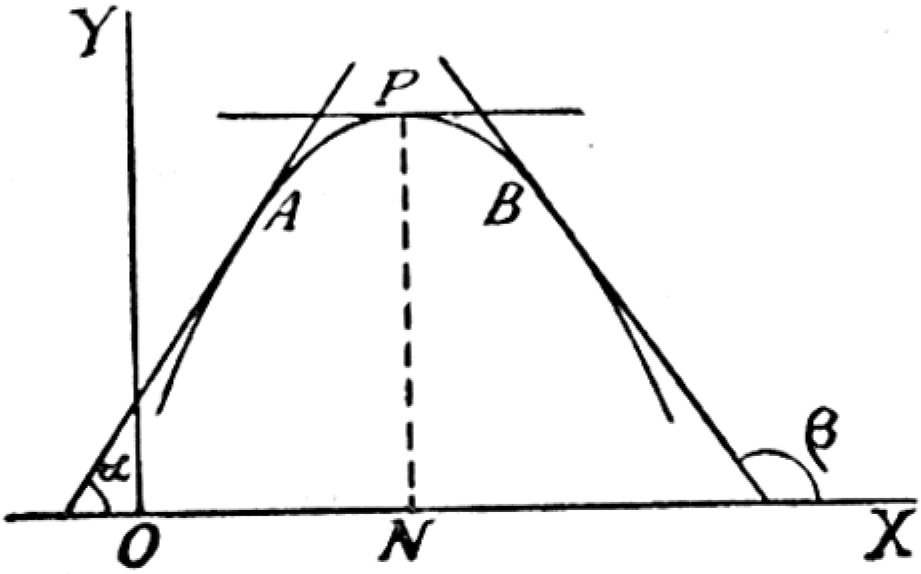
6. Show that $(x-1)^2(2x+7)$ decreases as x increases for values of x lying between -2 and 1.

7. Sketch the graph of $y=\sin x$ and $y=\cos x$ on the same axes from 0 to 2π and show that $y=\sin x$ increases or decreases where $\cos x$ is positive and negative respectively.

Maxima and Minima

7·21. Maximum Point. *A function is said to be a maximum at a point where it changes from an increasing to a decreasing one.*

At a point such as P where the function changes from an increasing to a decreasing one, the tangent is parallel to the x-axis and $\frac{dy}{dx}=0$.



At any point such as A just before P, the function being an increasing one, the tangent makes an acute angle (α) with the axis of x and $\frac{dy}{dx}$ is positive.

At any point such as B just after P, the function being a decreasing one, the tangent makes an obtuse angle (β) with the axis of x and $\frac{dy}{dx}$ is negative.

It is clear from the above facts that for a function $f(x)$, the ordinate at P is greater than all other ordinates in the immediate neighbourhood of P, on its left as well as on the right.

Thus $f(x)$ *is maximum at a point* $x=c$, *if the value of* $f(x)$ *at* $x=c$ *is algebraically greater than all values of* $f(x)$ *in the immediate neighbourhood. of* $x=c$ when x is slightly less than c *i.e.*, just on the left of P, $f(x)$ is an increasing function of x and, therefore, $f'(x)$ or $\frac{dy}{dx}$ is positive.

When x is slightly greater than c, *i.e.*, just on the right of P, $f(x)$ is a decreasing function of x and therefore, $f'(x)$ or $\frac{dy}{dx}$ is negative.

Hence $f'(x)$ changes from $+ve$ to $-ve$ as x changes from slightly $<c$ to slightly $>c$.

Also $f'(x)$ cannot change from $+ve$ to $-ve$ without taking the intermediate value 0 [$f'(x)$ is supposed to be continuous]. Therefore $f'(x)=0$ at $x=c$.

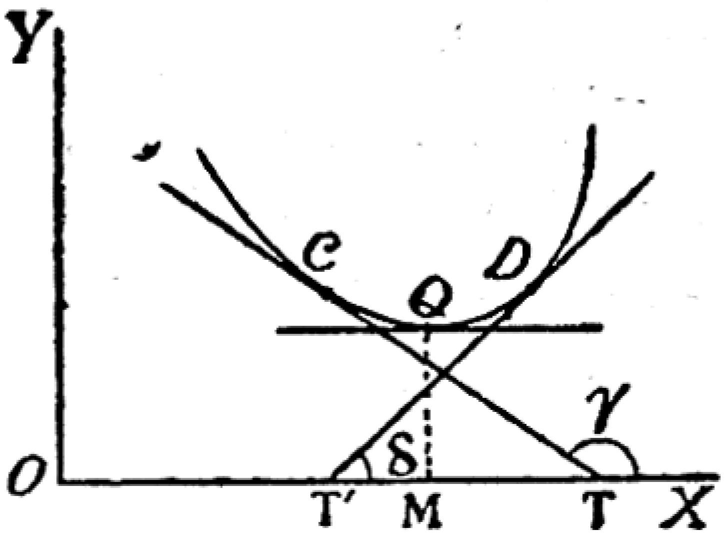
Hence for $f(x)$ to be a maximum at $x=c$,

(i) $f'(x)=0$ *at* $x=c$

and *(ii)* $f'(x)$ *changes from* $+ve$ *to* $-ve$ *as* x *changes from slightly* $<c$ *to slightly* $>c$.

7 22. Minimum Point. *A function is said to be a a minimum at a point where it changes from a decreasing to an increasing one.*

At a point such as Q where the function changes from a decreasing to an increasing one, the tangent is parallel to the axis of x and $\frac{dy}{dx}=0$.



At any point such as C just before Q, the function being a decreasing one, the tangent makes an obtuse angle (γ) with the x-axis and $\frac{dy}{dx}$ is negative.

At any point such as D, just after Q, the function being an increasing one, the tangent makes an acute angle (δ) with the x-axis and $\frac{dy}{dx}$ is positive.

It is clear from the above facts that,

$f(x)$ *is a maximum at a point* $x=d$ *if the value of* $f(x)$ *at* $x=d$ *is algebraically less than all values of* $f(x)$ *in the immediate neighbourhood of* $x=d$.

When x is slightly less than d *i.e.*, just on the left of Q, $f(x)$ is a decreasing function and, therefore, $f'(x)$ is negative.

When x is slightly greater than d *i.e.*, just on the right of Q, $f(x)$ is an increasing function and therefore $f'(x)$ is positive.

Hence $f'(x)$ changes from negative to positive as x changes from slightly $<d$ to slightly $>d$.

Also $f'(x)$ cannot change from $-ve$ to $+ve$ without taking the intermediate value 0 [$f'(x)$ is supposed to be continuous at $x=d$]. Therefore $f'(x)=0$ at $x=d$.

Hence for $f(x)$ to be a minimum at $x=d$,

(*i*) $f'(x)=0$ *at* $x=d$

and (*ii*) $f'(x)$ changes sign from $-ve$ to $+ve$ as x changes from slightly $<d$ to slightly $>d$.

7·23. Conditions for maxima and minima points :—

(*i*) $\frac{dy}{dx}=0$ *for maximum as well as for minimum points*, for which this is a necessary condition.

For the values of x obtained from this equation, the function may have either max. or min. values.

(*ii*) *For maximum values*, $\frac{dy}{dx}$ *must change from positive to negative.*

and (*iii*) *for minimum values*, $\frac{dy}{dx}$ *must change from negative to positive.*

Note. Maxima and minima points where dy/dx changes sign by passing through the value zero are also called **turning points;** for such points the curve either after increasing *turns* and begins to decrease, or after decreasing turns and begins to increase.

7·24. Point of Inflexion. If for some values of x obtained from the equatien $\frac{dy}{dx}=0$, $\frac{dy}{dx}$ does not change sign, the function is neither maximum nor minimum at these points.

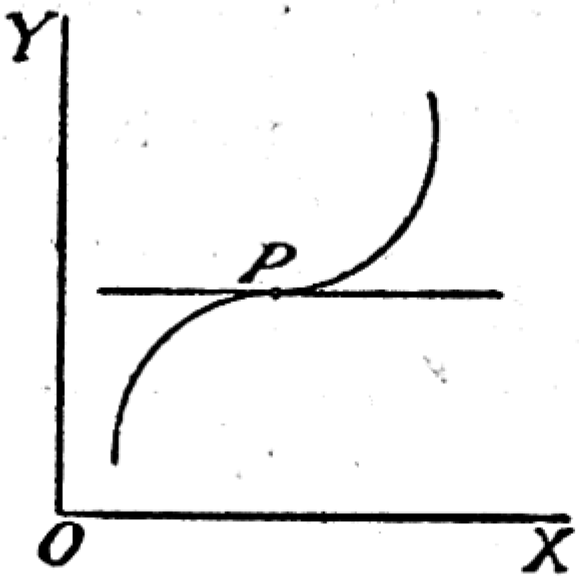


Fig (*i*)

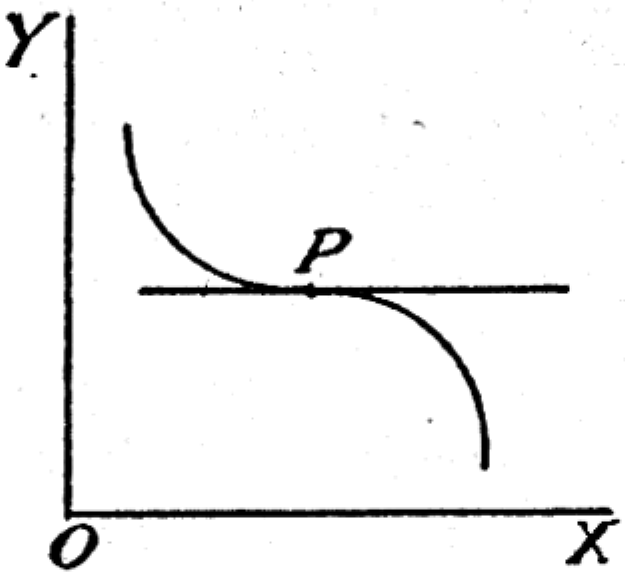


Fig (*ii*)

At such points as P in the figure, the tangent is parallel to the x axis but the curve does not change from an increasing to a decreasing one or *vice versa.* If the function is increasing just before P, it also remains increasing just after P Fig (*i*) and if it is decreasing just before P it remains decreasing just after P [Fig. (*ii*)].

Such points are known as *points of inflexion.*

It is not, however, necessary that at a point of inflexion the tangent should be parallel to the axis of x as is evident from the adjoining figure.

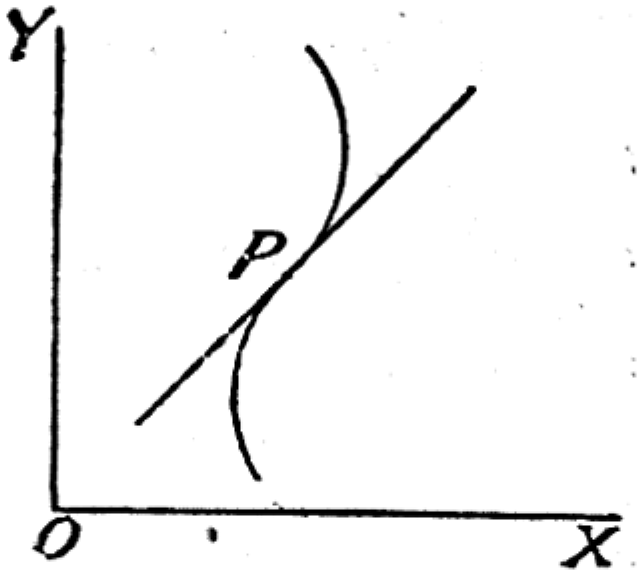


Fig. (*iii*)

Note. It may be observed that the curve is *convex upward* in the neighbourhood of a max. point and *concave upward* in the neighbourhood of min. a point. At a point of inflexion the curve changes from convex up to concave up [Fig. (*i*) and (*iii*)] or from concave up to convex up [Fig. (*ii*)].

Examples. Determine the max. and min. values of

$$x^5-5x^4+5x^3 \qquad \text{(P. U. 1946)}$$

Let $y=x^5-5x^4+5x^3$ ....(1)

Then $$\frac{dy}{dx}=5x^4-20x^3+15x^2=5x^2(x-1)(x-3) \qquad \text{..................(2)}$$

Equated to zero, it gives
(*i*) $x=1$, (*ii*) $x=3$ and (*iii*) $x=0$.

Now we proceed to test the nature of these points

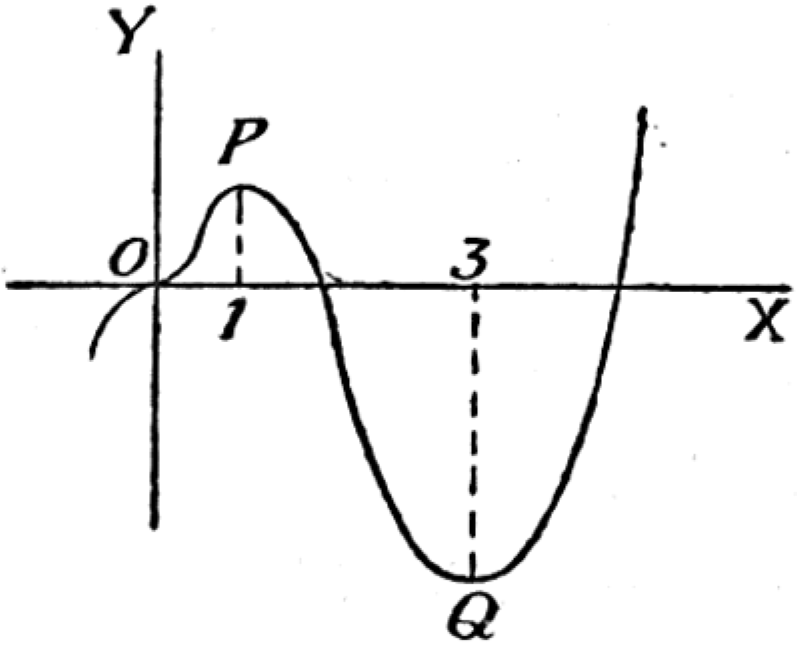
(*i*) If x is a little <1,

$\frac{dy}{dx}$ is $(+)(-)(-)$ *i.e.* positive.

If x is a little >1,

$\frac{dy}{dx}$ is $(+)(+)(-)$ *i.e.* negative.

$\therefore$ at $x=1, \frac{dy}{dx}=0$ and changes from $+ve$ to $-ve$.



Hence the function is max. at $x=1$ and by substituting $x=1$, in equation (1) we get the max. value $=1-5+5=1$

(*ii*) If x is a little <3, $\frac{dy}{dx}$ is $(+)(+)(-)$ *i.e.* negative.

If x is a little >3, $\frac{dy}{dx}$ is $(+)(+)(+)$ *i.e.* positive.

$\therefore$ at $x=3, \frac{dy}{dx}=0$ and changes from $-ve$ to $+ve$.

Hence the function is min. at $x=3$ and by the substitution of $x=3$ in (1), we get the min. value $=243-405+135=-27$.

(*iii*) If x is slightly <0, $\frac{dy}{dx}$ is $(+)(-)(-)$ *i.e.* positive.

If x is slightly >0, $\frac{dy}{dx}$ is $(+)(-)(-)$ *i.e.* positive.

Thus at $x=0$, $\frac{dy}{dx}$ does not change sign and the curve remains increasing just before and after this point and hence the point (0, 0) is a point of inflexion.

Exercise XX.

1. Find the point of inflexion and the max. and min. values of :—

(*i*) $12x^5-45x^4+40x^3+6.$ (*ii*) $3x^4-16x^3+18x^2.$

Draw a rough sketch of each and compare it with the figure drawn in the above worked example.

2. Find the max. and min. values of the expression given by $y=3x^5-5x^3+4$. If there is a point of inflexion, find it. (P. U. 1948)

3. Find the stationary points for the following curves :—

(*i*) $x^3-3x+6.$ (*ii*) $x^4-2x^2+12.$ (*iii*) $4x^2+y^2=16.$

(*iv*) $\frac{x}{1+x+x^2}.$ (*v*) $\frac{(x-1)(x-4)}{x}.$ (*v*) $\frac{(1-x)^3}{1-2x}.$

7·3. Second derivative test for maxima and minima. To distinguish between maxima and minima points of a function, we have to examine the change of sign in $\frac{dy}{dx}$. But this is not apparent at once in most cases. To facilitate the work we make use of the Second derivative.

At a max. point $\frac{dy}{dx}$ changes from +*ve* to −*ve*, so that $\frac{dy}{dx}$ itself is a decreasing function of x at that point.

Now since the derivative of a decreasing function is negative, therefore, the derivative of $\frac{dy}{dx}$ *i.e.* $\frac{d^2y}{dx^2}$ is negative at a max. point.

At a min. point $\frac{dy}{dx}$ changes from −*ve* to +*ve* so that $\frac{dy}{dx}$ itself is an increasing function of x at that point, hence its derivative *i.e.* $\frac{d^2y}{dx^2}$ must be positive at the point.

Hence $\frac{d^2y}{dx^2}$ *is* $-ve$ *for a max. point and* $+ve$ *for a min. point.*

This test will usually be found easier to apply than the test by discussion of the change of sign in $\frac{dy}{dx}$.

However, it cannot be relied upon entirely, because it may happen that for some values of x obtained from the equation $\frac{dy}{dx}=0$, $\frac{d^2y}{dx^2}$ may not be $+ve$ or $-ve$ but may be zero. In that case this test fails and the nature of the stationary values must be decided by considering the change of sign in $\frac{dy}{dx}$.

Note. We may also proceed as follows :—

When $\frac{dy}{dx}=0$, $\frac{d^2y}{dx^2}=0$ but $\frac{d^3y}{dx^3} \neq 0$, for some value of x, the function is neither a max. nor a min. but has a point of inflexion at that point.

If $\frac{d^3y}{dx^3}$ also vanishes for the value of x under consideration, the function will be max. if $\frac{d^4y}{dx^4}$ is negative and min. if it is positive.

If $\frac{d^4y}{dx^4}=0$ also, proceed further on until either some *odd differential co-efficient* (other than the first) does not vanish in which case there is a point of inflexion, or some *even differential co-efficient* is negative or positive, in which case the function has either a maximum or a minimum value.

The proof of this statement is beyond the scope of book.

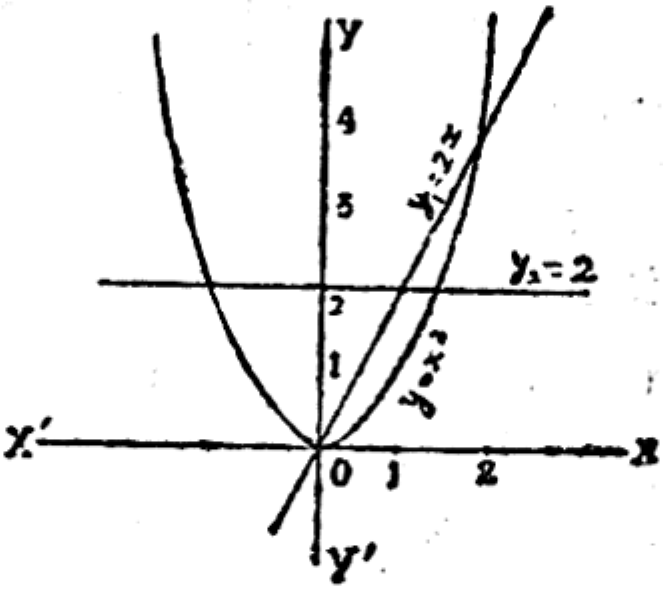
Examples

Ex. 1. *Show graphically that the parabola $y=x^2$ has a min. value at the origin.*

Here $y_1=\dfrac{dy}{dx}=2x$, $y_2=\dfrac{d^2y}{dx^2}=2$.

The graphs of the functions $y=x^2$, $y_1=2x$ and $y_2=2$ are drawn with reference to the same axes.

$y=x^2$ is the parabola, $y_1=2x$ is the straight line passing through the origin and having its slope equal to 2, and $y_2=2$ is the straight line parallel to the x-axis at a constant distance 2 from it.



At the origin the graph of $y_1=2x$ changes from $-ve$ to $+ve$.

Also y_2 being equal to 2 is positive.

Hence the curve is minimum at the origin, its min. value being zero.

Ex. 2. Find the stationary points for the curve

$$y=x^5-5x^4+5x^3$$

and indicate them by drawing a rough sketch.

Here $y_1=\dfrac{dy}{dx}=5x^4-20x^3+15x^2=5x^2(x-1)(x-3)$;

$$y_2=\frac{d^2y}{dx^2}=10x(2x^2-6x+3)$$

$y_1=0$ when $x=1$, $x=3$ and $x=0$.

For $x=1$, $y_2=-10$; $\therefore$ it gives a max. point.

For $x=3$, $y_2=90$; $\therefore$ it gives a min. point.

For $x=0$, $y_2=0$, but y_1 does not change sign; it, therefore, gives a point of inflexion.

It may be seen that the third derivative $y_3 \neq 0$ for $x=0$. This also shows that $x=0$ gives a point of inflexion.

This is the same example as worked in Art. 7·24 and it shows that test by the second derivative is much easier than the test by consideration of the sign of y_1. For the rough sketch see Art. 7·24.

Ex. 3. Find the turning points and the values of y for these points for the curve

$$y=3x^5-25x^3+60x.$$

Let $y=3x^5-25x^3+60x$

$$\frac{dy}{dx}=15x^4-75x^2+60$$

$$=15(x^2-4)(x^2-1),$$

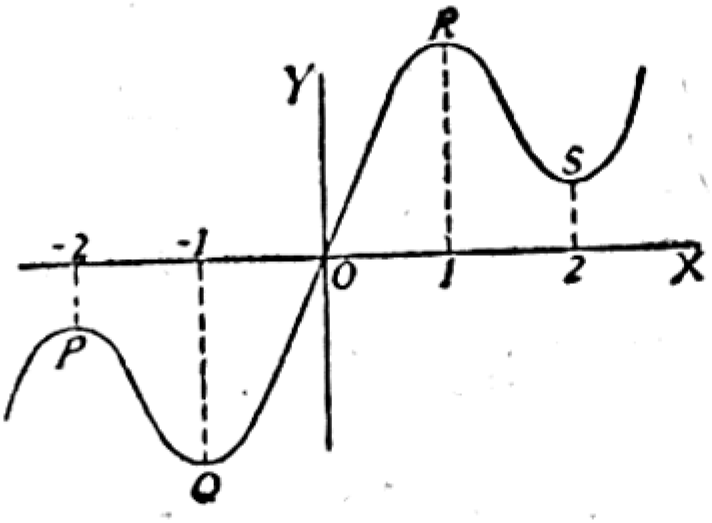
which equated to zero gives

$$x=-2,\ x=-1.$$

$$x=1,\quad x=2,$$

$$\frac{d^2y}{dx^2}=60x^3-150x$$

$$=30x(2x^2-5).$$



x	$\frac{d^2y}{dx^2}$	value of y
-2	$-ve$	Max. value $=-16$
-1	$+ve$	Min. ,, $=-38$
$+1$	$-ve$	Max. ,, $=\ 38$
$+2$	$+ve$	Min. ,, $=\ 16$

Notice that (*i*) max. and min. values occur alternately and (*ii*) that the min. value 16 at $x=2$ is greater than the max. value -16 at $x=-2$.

This illustrates the fact that the relative magnitudes of the values of a function is not the criterion for distinguishing between its max. and min, values.

Ex. 4. Examine the curve $y=\sin x(1+\cos x)$ for its stationary points for values of x lying between 0 and 2π.

We have $y=\sin x(1+\cos x)=\sin x+\frac{1}{2}\sin 2x$. ... ...(1)

$$\therefore \quad y_1=\frac{dy}{dx}=\cos x+\cos 2x=2\cos^2 x+\cos x-1$$

$$=(2\cos x-1)(\cos x+1) \quad \dots \quad \dots \quad \dots(2)$$

$$y_2=\frac{d^2y}{dx^2}=-\sin x-2\sin 2x,$$

$$y_3=\frac{d^3y}{dx^3}=-\cos x-4\cos 2x.$$

By (2), $y_1=0$ when $\cos x=\frac{1}{2}$ which gives $x=\pm\frac{1}{3}\pi$,
and $\cos x=-1$ which gives $x=\pi$.

For $x=\frac{1}{3}\pi$, y_2 is $-ve$, $\therefore$ y is max., and by (1), the max. value$=3\sqrt{3}/4$.

For $x=-\frac{1}{3}\pi$, y_2 is $+ve$, $\therefore$ y is min., the min. value $=-3\sqrt{3}/4$.

For $x=\pi$, $y_2=0$ but $y_3\neq 0$, $\therefore$ it gives a point of inflexion.

Exercise XXI

1. Examine the following functions for their stationary values :—

(*i*) $x^5-5x^4+5x^3-1$. (P.U. 1951)
(*ii*) $2x^3-9x^2-24x-20$. (P.U. 1950)
(*iii*) $2x^3-21x^2+36x-20$. (P.U. 1949)
(*iv*) $(x-3)^5(x+1)^4$. (P.U. 1945)
(*v*) $3x^5-53x^3+135x$.
(*vi*) $(x-1)(x-2)^2(x-3)^3$. (D.U.)

2. Find the stationary values of the following curves and determine whether each is a max., min. as a point of inflexion :—

(*i*) $x+\cos x$. (*ii*) $3\sin^2 x+4\cos^2 x$ (P.U. 1952) (*iii*) $\sin x-\sin^2 x$. (*iv*) $\dfrac{\sin x}{2+\cos x}$.

(*v*) $\dfrac{\sin x}{5+3\cos x}$ (*vi*) $2\tan x-\tan^2 x$ (P.U.)

(*vii*) $\sin x\cos^3 x$ (P.U.) (*viii*) $\sin x\cos 2x$.

(*ix*) $(\cos x+2\sin x)^2$ (P.U.)

(*x*) $\sin x\cos x/(\sin x+\cos x)$ (P.U.)

3. The range, R, of a criket ball projected on a horizontal plane is given by the formula

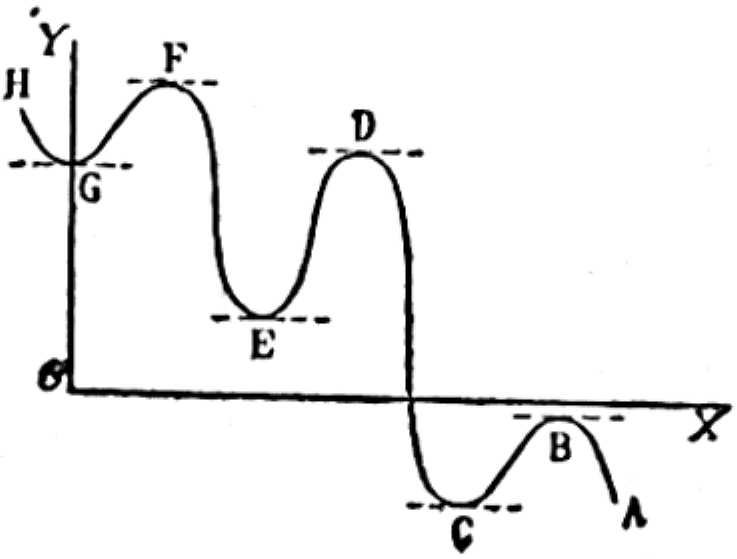
$$gR=2V^2\sin\theta\cos\theta,$$

where V and g are constants ; find the value of θ when R is a maximum. (D.U.)

4. If $u=f(x)$ be a maximum for some values of x show that $\dfrac{1}{u}$ is a minimum for the same value of x. (P.U.)

5. The graph of $y=\dfrac{ax+b}{(x-1)(x-4)}$ has a turning point at P(2, −1). Determine the values of a and b and show that y is a maximum at P. (P.U. 1951)

7·4. Maxima and Minima distinguished from greatest and least: The words "maximum" and "minimum" are not used precisely in the same sense as "greatest, and "least" ; they only signify; *greatest or least in the immediate neighbourhood.* Notice that in the figure, the minimum value at E is greater than the maximum value at B and that there are several maximum and minimum values for the same function. This is explained by the fact that by the maximum and minimum values we do not mean the greatest and the least values of the function. If the curve always slopes upwards and if we extend the curve to the left



of H, we can find a value of x that makes $f(x)$ greater than any of the maximum values, and similarly to the right of A, $f(x)$ may become smaller than any of the minimum values.

By the function being maximum at any point say at D, we simply mean that the value of the function at D, is greater than what it is at other point in the immediate neighbourhood of the point D, either to the left or to the right to it.

Similarly at the minimum point E, the value of the function is less than that what it is at any other point just near E.

"Maxima are like mountain tops, and minima like valley bottoms. A hill-top is higher than surrounding points but there may be a higher mountain near."

7·41 Maxima and minima values occur alternately.

In the figure, the function is a maximum at F, and again at D. Just after the point F, the ordinate of the function begins to decrease and just before the point D, the function is increasing. Hence between F, and D, the function changes from a decreasing to an increasing one, and, if the function is continuous, this cannot take place unless the function passes through *at least* one minimum value. Thus between two maxima, one minimum at least must exist and similarly between two minima at least one maximum must exist. Hence maxima and minima occur alternately in a continuous curve. (*See Ex. 3 Art. 7·3*)

Ex. 1. How do you explain the fact that though

$$y=2x^3-9x^2+12x+1$$

can be made as large as we please by taking x to be large enough, the function has a maximum value when $x=1$. (P.U.)

Ex. 2. Explain what you mean by the maximum and minimum values of a function. Distinguish them clearly from the greatest possible and the least possible values of the function.

7·5 Additional Examples. A few examples of a general character are given below :—

Ex. 1. Find the max. and min. values of

$$y=a \sin x+b \cos x.$$

First Method. *By Calculus* :

$$y_1=\frac{dy}{dx}=a \cos x-b \sin x,$$

$$y_2=\frac{d^2y}{dx^2}=-(a \sin x+b \cos x)$$

$$y_1=0 \text{ when } \tan x=\frac{a}{b},$$

so that $\sin x=\pm\frac{a}{\sqrt{a^2+b^2}}$; $\cos x=\pm\frac{b}{\sqrt{a^2+b^2}}$

y_2 is negative for positive values of $\sin x$ and $\cos x$.

$\therefore$ the max. value of $y=a.\frac{a}{\sqrt{a^2+b^2}}+b.\frac{b}{\sqrt{a^2+b^2}}$

$$=\sqrt{a^2+b^2}$$

Again y_2 is the positive for negative values of y,

hence the min. value of $y=a\left(-\frac{a}{\sqrt{a^2+b^2}}\right)+b\frac{-b}{\sqrt{a^2+b^2}}$

$$=-\sqrt{a^2+b^2}.$$

Second method. *By Trigonometry.*

Let $a=r \cos\theta$, $b=r \sin\theta$, so that

$$r=\pm\sqrt{a^2+b^2} \text{ and } \tan\theta=b/a$$

$$\therefore\ y=r(\sin x \cos\theta+\cos x \sin\theta)$$

$$=r \sin(x+\theta)$$

$$=\pm\sqrt{a^2+b^2}\sin(x+\tan^{-1}b/a)$$

Now as the greatest value of sine of an angle is unity in magnitude, therefore, the greatest and least values of y are $\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$ respectively.

Note. In this case, the greatest and least values of the function are also the maximum and minimum values.

Ex. 2. Find the minimum values of

$$a^2/\cos^2 x + b^2/\sin^2 x.$$

Let $y = a^2/\cos^2 x + b^2/\sin^2 x = a^2\sec^2 x + b^2\operatorname{cosec}^2 x$ (1)

$\therefore$ $y_1 = 2(a^2 \sec^2 x \tan x - b^2 \operatorname{cosec}^2 x \cot x)$ ...... (2)

$y_1 = 0$ when $a^2 \sec^2 x \tan x = b^2 \operatorname{cosec}^2 x \cot x$

i e. $a^2 \sin^2 x \quad \tan^2 x = b^2 \cos^2 x$

or $\tan^4 x = b^2/a^2$ $\quad\therefore\quad \tan^2 x = \pm b/a.$

Now $y_1 = 2(a^2 \tan x + a^2 \tan^3 x - b^2 \cot x - b^2 \cot^3 x)$

$$\therefore \quad y_2 = 2(a^2 \sec^2 x + 3a^2 \tan^2 x \sec^2 x + b^2 \operatorname{cosec}^2 x + 3b^2 \cot^2 x \operatorname{cosec}^2 x)$$

which is positive for the positive value b/a of $\tan^2 x$

$\therefore$ y is min. when $\tan^2 x = \dfrac{b}{a}$,

so that $\sec^2 x = \dfrac{a+b}{a}$, $\operatorname{cosec}^2 x = \dfrac{a+b}{b}$.

Hence the min. value of y

$$= a^2\left(\frac{a+b}{a}\right) + b^2\left(\frac{a+b}{b}\right) = (a+b)^2.$$

Exercise XXII

1. Find the max. and min. values of.

(*i*) $\sin x + \cos x$. (P.U. 1945).

(*ii*) $3 \sin x - 4 \cos x$.

(*iii*) $a \sin^2 x + b \cos^2 x$. (P.U.)

(*iv*) $\cos^2 x/a^2 + \sin^2 x/b^2$. (P·U.)

2. Prove that $a \sec\theta + b \operatorname{cosec}\theta$ is a minimum when $\tan \theta = (b/a)^{1}$

3. (*a*) Show that the function $\sin^p\theta \cos^q\theta$ attains its maximum at $\theta = \alpha$ where $\tan \alpha = \sqrt{p/q}$.

(*b*) show that $\sin^2 x \cos^3 x$ is maximum when

$$x = \tan^{-1}\sqrt{\tfrac{2}{3}}.$$

4. Examine the function $\dfrac{x^2+x+1}{x^2-x+1}$ for max. and min. values. (D.U.)

Verify the answer algebraically.

5. Investigate for max. and min. values :—

(*i*) $\sin x + \cos 2x$. (*ii*) $\sec x + \cos x$.

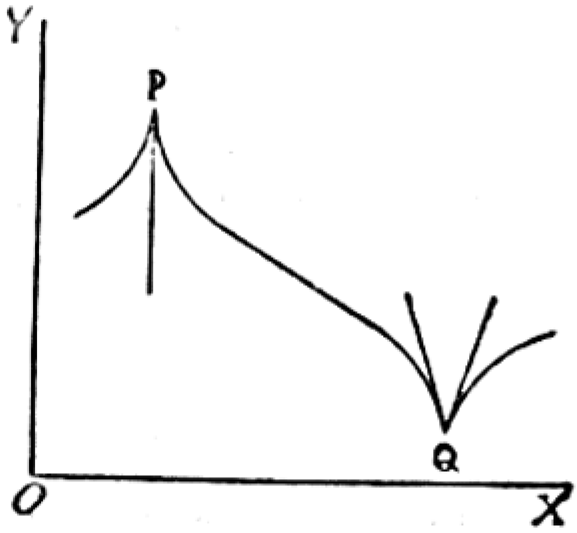
(*iii*) $x + \tan x$. (*iv*) $2\tan x - \tan^2 x$.

(*v*) $\sin^2\theta + \cos\theta$.

7·6 Exceptional cases of maxima and Minima.

The methods for finding maxima and minima values of a function given in Arts. 7·23. and 7·3. are sufficient if the *function* and its *derivative* are continuous.

Let us now consider what takes place at points such as P and Q in the accompanying figure.



At P the function changes from an increasing to a decreasing one, while at Q, it changes from a decreasing to an increasing one. Hence P is a maximum point and Q is a minimum point.

At the point P, the two branches have the same tangent (parallel to OY) the slope of of which is infinite (tan 90°); at the point Q, the two branches have different tangents. At P the derivative does not exist and at Q the derivative is discontinuous.

Points such as P and Q are extremely unusual and require special methods which are beyond the scope of this book.

The maximum or minimum value at a point such as P can be obtained by finding the values of x which make the derivative, $\frac{dy}{dx}$ infinity or $\frac{dx}{dy}=0$

Examples

1. Show that the origin is a min. point on the curve $y^3=x^2$.

In this case, $y=x^{\frac{2}{3}}$ and $\frac{dy}{dx}=\frac{2}{3}x^{-\frac{1}{3}}$

There is no finite value which makes the derivative $dy\,dx$ zero, but $x=0$ makes it infinite, and dy/dx changes sign from $-ve$ to $+ve$ at this point. Also y is never negative, hence the value 0 at $x=0$ is a minimum.

2. (*i*) Show that $y^3=(x-1)^2$ has a min. value at $x=1$.

(*ii*) Show that in the curve $(y+3)^3=(x-2)^2$, y is minimum at $x=2$.

3. Show that the origin is a max. point on the curve $x^2+y^3=0$.

4. Show that for the curve $y=1-x^{\frac{2}{3}}$, dy/dx does not exist and y has the max. value 1 at the point $x=0$.

Problems on Maxima and Minima

7·7 One of the principal applications of the calculus is to find under what conditions the value of a function is a maximum or a minimum. This often finds its utility in problems of engineering, where it is most desirable to know what conditions will make the cost of working minimum or will make the efficiency maximum.

Besides engineering problems, most of the problems on maxiuma and minima met with in Algebra and Geometry can be very conveniently solved by the aid of calculus.

Examples

7·71 Algebric and geometrical problems

Ex. *Find the least value of the sum of a positive number and its reciprocal.*

Let x denote any positive mumber and let the sum of this mumber and its reciprocal be y, then

$$y=x+\frac{1}{x};\ y_1=\frac{dy}{dx}=1-\frac{1}{x^2};\quad y_2=\frac{d^2y}{dx^2}=\frac{2}{x^3},$$

$y_1=0$ when $x^2-1=0$ *i.e.* $x=\pm 1$

As we are concerned with the positive value, we have $x=1$ for which y_2 is + ve.

Hence y is min. for $x=1$, its min. value being 2.

Ex. 2 *Given the perimeter of a rectangle, show that*

(*i*) *its area is a maximum*

and (*ii*) *its diagonal a minimum*

when it is a square.

Let x and y be the lengths of the sides of the rectangle, A its area and Z, the length of its diagonal.

Then the perimeter of the rectangle

$$=2x+2y=2c \text{ (say)}$$

$\therefore \quad x+y=c \qquad$ or $y=c-x \qquad$ ....(i)

(i) In the first case, $\quad A=xy=x(c-x)=cx-x^2$

$$\frac{dA}{dx}=c-2x, \quad \frac{d^2A}{dx^2}=-2$$

$dA/dx=0$ gives $x=\frac{1}{2}c$, d^2A/dx^2 is always $-ve$.

$\therefore$ A is max. when $x=\frac{1}{2}c$, then by (1) $y=\frac{1}{2}c$

Hence the area is max. when the sides of the rectangle are equal *i.e.* the rectangle is a square.

(ii) In this case $Z=\sqrt{x^2+y^2}$ or $Z^2=x^2+y^2$
$$=x^2+(c-x)^2.$$

$$\therefore \; 2Z\frac{dZ}{dx}=2x-2(c-x)=4x-2c \text{ or } Z\frac{dZ}{dx}=2x-c \quad ...(2)$$

$\therefore$ $dZ/dx=0$ gives $x=\frac{1}{2}c$, for which $y=\frac{1}{2}c$, so that the rectangle is a square.

Differentiating (2), we have,

$$\left(\frac{dZ}{dx}\right)^2+Z\frac{d^2Z}{dx^2}=2,$$

$\therefore \; \dfrac{d^2Z}{dx^2}$ is $+ve$, since $\dfrac{dZ}{dx}=0$ and Z is $+ve$.

Hence Z has a min. value for $x=y=\frac{1}{2}c$.

Ex. 2. *The sum of the perimeters of a circle and a square is l. Show that when the sum of their areas is least the side of the square is equal to the diameter of the circle.*

Let x be the side of the square and r, the radius of the circle, then by the question, we have

$$4x+2\pi r=l. \qquad ... \qquad ... \qquad ...(1)$$

Let y be the sum of the areas of the square and the circle, then $\qquad y=x^2+\pi r^2 \qquad ... \qquad .. \qquad ...(2)$

Differentiating (1) and (2) with respect to x, we have

$$\text{From (1) } 4+2\pi\frac{dr}{dx}=0 \qquad i.e.\ \pi\frac{dr}{dx}=-2 \qquad \ldots(3)$$

and from (2) $\dfrac{dy}{dx}=2x+2\pi r\dfrac{dr}{dx}=2x-4r\ldots$ ...[by (3)]

which equated to zero gives x, the side of the square $=2r$, the diameter of the circle.

Also $\dfrac{d^2y}{dx^2}=2-4\dfrac{dr}{dx}=2+\dfrac{8}{\pi}$ ... ...[by (3)]

which being $+ve$, shows that y is min. for $x=2r$.

Exercise XXIII.

1. Find the fraction which exceeds its square by the greatest amount.

2. If $x+y=5$, find the maximum value of x^3y^2. (D.U.)

3. Divide 18 into two parts such that their product is a max. and the sum of their squares is a min.

4. A line of length 8 inches is divided into two parts AP and PB by a point P, find the position of P if AP^2+BP^2 is minimum. (P.U. 1950)

5. A gutter of rectangular section is to be made by bending into shape a rectangular strip of copper. Show that the capacity of the gutter will be greatest if its width is twice its depth.

6. A rectangular enclosure containing 162 sq. ft. is to be fenced off; one side of the rectangle is to be along a stream where no fence is necessary. Find the dimensions of the enclosure which will require the least amount of fencing material.

7. A rectangular plot of ground is to be fenced off and divided into three parts by fences parallel to one of the sides; what should be the dimensions in order that as large an area as possible may be enclosed with 2000 ft. of fencing material.

8. The sum of the length and girth of a package to go by parcel post must not exceed 84 inches. Find the dimensions of the largest acceptable rectangular box with square ends.

If the parcel is made up in the shape of a circular cylinder, show that its length is 28 inches and the girth 56 inches.

9. A church window consists of a semicircle surmounting a rectangle. If the perimeter of the window is fixed, show that its area is maximum when $h=r$. r being the radius of the circle and h, the height of the rectangle.

If the perimeter of the window is 35·7 ft, show that it will admit the greatest amount of light if $h=r=5$ ft. ($\pi=3{\cdot}14$).

10. A length of wire 71·4 ft. long is cut into two portions which are bent into the shapes of a circle and a square. Find the radius of the circle and the side of the square when the sum of the areas is a minimum. ($\pi=3{\cdot}14$).

11. A cylindrical cistern is to be constructed with open top and max. volume. If its total surface area is 20π square ft., find its dimensions.

12. It is required to form equal circular cylindrical closed cans of max. volume from a thin sheet of metal; find the relation between the height and the radius of the base of cans so as to obtain the best result. (P.U. 1952)

13. Find the relation between the height and the radius of the base of a right circular cylinder of constant volume so that the surface may be minimum.

14. (*i*) The slant side of a cone is s, find the base radius and the height so that the volume may be the greatest.

(*ii*) Show that the semi-vertical angle of the cone of max. volume and of given slant side is $\tan^{-1}\sqrt{2}$. (P.U.)

15. A circular filter paper of radius 3 inches is to be folded into a conical filter. Find the radius of the base of the filter if it has a max. capacity.

16. (*i*) A rectangular sheet of metal is 20 inches by 10 inches. Four equal squares, side x inches, are removed from each corner, the edges left are turned up to be perpendicular to the base, and so form a box. Show that it holds $(4x^3-60x^2+200x)$ cubic inches, and find the value of x that makes this volume the greatest.

(*ii*) A tray is to be made from a sheet 12 inches square by cutting squares from its corners and then bending up its sides. Find the diamensions of the tray of max. capacity.

17. (*i*) Given the perimeter of a rectangle, find for what shape is the diagonal least.

(*ii*) Given two sides of a triangle, when is the area maximum ?

(*iii*) Show that the triangle of the greatest area with given base and vertical angle is isosceles.

18. (*i*) Find the right-angled triangle of maximum area that can be constructed on a given line as hypotenuse.

(*ii*) Given the sum of the lengths of the hypotenuse and another side of a right-angled triangle, show that the area of the triangle is maximum when the angle between these sides is 60°.

(*iii*) The perimeter of a right-angled triangle is constant. Prove that its area will be a maximum when it is isosceles.

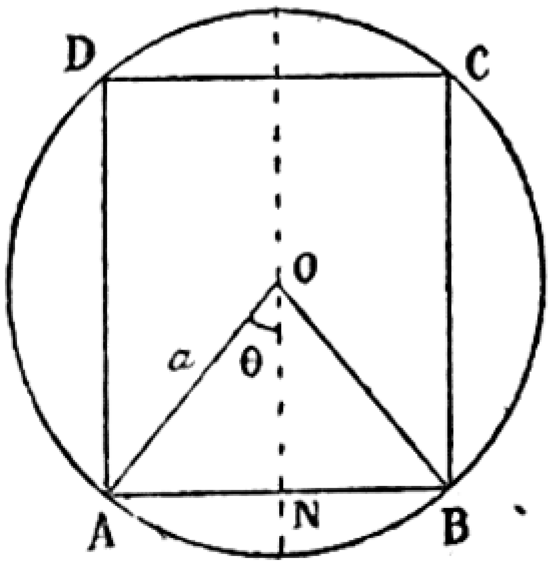
19. Among all circular sectors with a given perimeter, find the one which has the greatest area.

20. Given the total surface of a cone show that when the volume is a max., the semi-vertical angle of the cone $\sin^{-1}\dfrac{1}{\sqrt{3}}$.

7·72 Many problems in maxima and minima may be solved very neatly by expressing the function in terms of trigonometric functions of an angle.

Ex. 1. *Find the rectangle of max. area that can be inscribed in a given circle.*

Let ABCD be any rectangle inscribed in a circle of radius a and centre O, ON perp. on the side AB and $\angle AON=\theta$



Then $AN=a \sin\theta$,

and $ON=a \cos\theta$

$\therefore \quad AB=2AN=2a \sin\theta$,

$BC=2ON=2a \cos\theta$.

Let y denote the area of the rectangle, then $y=AB.BC$ $=2a \sin \theta .\ 2a \cos \theta=2a^2 \sin 2\theta$.

$\therefore \dfrac{dy}{d\theta}=4a^2 \cos 2\theta$, which is zero when $2\theta=90°$ or $\theta=45°$.

Hence for y to be max. $\angle AOB=90°$ *i e.* the rectangle is a square whose each side$=2a \sin 45°=a\sqrt{2}$.

Cor. The perimeter P of the rectangle

$$=2AB+2BC=4a \sin \theta+4a \cos \theta.$$

$\therefore \quad dP/d\theta=4a (\cos \theta-\sin \theta)$ is zero when $\tan \theta=1$ or $\theta=45°$

Thus *the rectangle of max. are inscribed in a circle is also the rectangle of max. perimeter.*

Note. $d^2y/d\theta^2=-8a^2 \sin 2\theta$ is$-ve$ for $\theta=45°$ and thus area is max. and not min. but the nature of the problem is such that it does not admit of a minimum value and hence the discussion of the sign of the second derivative is unnecessary in such cases.

Ex. 2. *Find the right circular cylinder of max. volume that can be inscribed in a sphere of radius a.*

Let the above figure (of Ex. 1) represent the section of the sphere and inscribed cylinder by a plane through the axis of the cylinder, then.

r, the radius of the cylinder$=AN=a \sin \theta$,

h, the height of the cylinder$=2\ ON=2a \cos \theta$.

Let V denote the volume of the cylinder, then

$$V=\pi r^2h=\pi a^2 \sin^2\theta . 2a \cos \theta=2\pi a^3 \sin^2\theta \cos\theta \text{....(1)}$$

$$\frac{dV}{d\theta}=2\pi a^3 (2 \sin\theta \cos^2\theta-\sin^3\theta)$$

this is zero when $\sin \theta\ (2 \cos^2\theta-\sin^2\theta)=0$

Rejecting $\sin\theta=0$, for which the cylinder does not exist, we get

$$2\cos^2\theta=\sin^2\theta, \text{ or } \tan^2\theta=2, \quad \therefore \cos\theta=\frac{1}{\sqrt{3}}, \ \sin\theta=\sqrt{\tfrac{2}{3}}$$

Hence from (1) the max. volume $=2\pi a^3 . \dfrac{2}{3} . \dfrac{1}{\sqrt{3}}$

$$=4\pi\, a^3/3\sqrt{3}.$$

The height h of the cylinder of max. volume

$$=2a\cos\theta=2a/\sqrt{3}.$$

and its diameter $d=2r=2a\sin\theta=2a\sqrt{\tfrac{2}{3}}$;

so that $\qquad d=\sqrt{2}h.$

Note. The discussion of the sign of the second derivative is unnecessary. But it may be noted that the second factor $\sin\theta=0$ gives the min. value which reduces the cylinder to a line coinciding with a diameter of the circle.

Ex. 3. *Find the right circular cylinder inscribed in a sphere of radius a so that its convex surface is a maximum.*

Let S denote the area of the convex surface,

we have $S=2\pi\, r.h=2\pi a\sin\theta.2a\cos\theta=2\pi a^2\sin 2\theta$

This is max. when $\sin 2\theta=1$ or $2\theta=90°$, *i.e.* $\theta=45°$.

Hence the max. value of $S=2\pi a^2$.

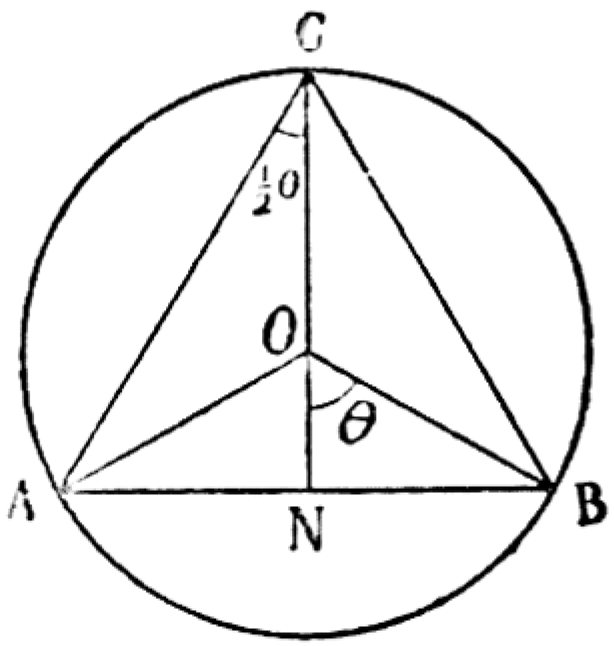
d, the diameter for max. $S=2a\sin\theta=2a/\sqrt{2}=a\sqrt{2}$.

h, the height for max. $S=2a\cos\theta=a\sqrt{2}$.

So that in this case the diameter of the cylinder is equal to its height.

Ex. 4. *Isosceles triangles are inscribed in a circle of given radius. Show that the triangle of max. area is equilateral.*

Let a be the radius of the circle, O its centre, AC, BC the equal sides of the isosceles triangle inscribed in the circle, and CN its altitude, then by symmetry of the figure O lies on CN and AN=NB.



Let $\angle BON=\theta$,

then $\qquad BN=a\sin\theta.$

and $\qquad CN=CO+ON,$

$$=a+a\cos^2\theta.$$

If $\triangle$ denotes the area of the triangle, then

$$\Delta=\text{AN}.\,\text{CN}=a\sin\theta(a+a\cos\theta)$$
$$=a^2(\sin\theta+\sin\theta\cos\theta)$$
$$\frac{d\Delta}{d\theta}=a^2(\cos\theta+\cos^2\theta-\sin^2\theta)$$
$$=a^2(2\cos^2\theta+\cos\theta-1)$$

This is zero when $a^2(2\cos\theta-1)(\cos\theta+1)=0$

Rejecting the solution $\cos\theta+1=0$ for which there is no triangle, we have

$$2\cos\theta-1=0 \text{ i.e. } \cos\theta=\tfrac{1}{2}, \quad \therefore\ \theta=60°,$$

so that $\angle\text{AOB}=120°$ and $\angle\text{ACB}=60°$.

Hence for max. area the triangle is equilateral.

Ex. 5. *Find the right circular cone of (i) the greatest volume (ii) the greatest curved surface that is inscribed in a sphere of radius a.*

In the above figure (of Ex. 4) let ACB represent the section of the sphere and the cone by a plane through the axis of the cone, then

r, the radius of the cone$=\text{AN}=a\sin\theta$.

h, the height of the cone$=\text{CN}=\text{CO}+\text{ON}=a+a\cos\theta$.

(*i*) Let V denote the volume of the cone, then

$$\text{V}=\tfrac{1}{3}\pi r^2h=\tfrac{1}{3}\pi a^2\sin^2\theta(a+a\cos\theta)$$
$$=\tfrac{1}{3}\pi a^3(1+\cos\theta-\cos^2\theta-\cos^3\theta).$$
$$\therefore\ \frac{d\text{V}}{d\theta}=\tfrac{1}{3}\pi a^3(-\sin\theta+2\sin\theta\cos\theta+3\cos^2\theta\sin\theta)$$
$$=\tfrac{1}{3}\pi a^3\sin\theta(3\cos\theta-1)(\cos\theta+1).$$

This is zero when $\cos\theta=\tfrac{1}{3}$, $\sin\theta=0$, $\cos\theta+1=0$ being rejected.

Hence V is max. when $\cos\theta=\tfrac{1}{3}$ and $\sin^2\theta=1-\tfrac{1}{9}=\tfrac{8}{9}$.

$\therefore$ the max. value of $\text{V}=\tfrac{1}{3}\pi a^3\cdot\tfrac{8}{9}(1+\tfrac{1}{3})=\tfrac{32}{81}\pi a^3$.

h for max. $\text{V}=a+a\cos\theta=a+\tfrac{1}{3}a=\tfrac{4}{3}a$.

d, the diameter of the cylinder of max. V

$$=2r=2a\sin\theta=\frac{4a\sqrt{2}}{3}$$

so that $d=\sqrt{2}h$.

(*ii*) $\angle OCA = \angle OAC = \frac{1}{2}\theta$. $\therefore$ $AC = 2a \cos \frac{1}{2}\theta$.

Let S denote the area of the curved surface,

then $S = \frac{1}{2} \cdot 2\pi AN \cdot AC = \pi a \sin\theta \, 2a \cos\frac{1}{2}\theta$

$$= 4\pi a^2 \sin\tfrac{1}{2}\theta \cos^2\tfrac{1}{2}\theta.$$

$$\therefore \frac{dS}{d\theta} = 4\pi a^2[\tfrac{1}{2}\cos^3\tfrac{1}{2}\theta - 2\cdot\tfrac{1}{2}\sin^2\tfrac{1}{2}\theta \cos\tfrac{1}{2}\theta)$$

$$= 2\pi a^2 \cos\tfrac{1}{2}\theta(\cos^2\tfrac{1}{2}\theta - 2\sin^2\tfrac{1}{2}\theta).$$

This is zero when either $\cos\frac{1}{2}\theta = 0$ *i.e.*, $\theta = \pi$ for which the cone does not exist,

or $\tan^2\frac{1}{2}\theta = \frac{1}{2}$ which gives

$$\sin\tfrac{1}{2}\theta = \frac{1}{\sqrt{3}} \quad \text{and} \quad \cos\tfrac{1}{2}\theta = \sqrt{\tfrac{2}{3}}$$

Hence max. $S = 4\pi a^2 \cdot \frac{1}{\sqrt{3}} \cdot \frac{2}{3} = 8\pi a^2/3\sqrt{3}$.

Exercise XXIV.

1. A rectangle is inscribed in a semi-circle of radius a so that one side lies on the diameter and the other two corners on the circular arc. Find its maximum area.

2. Find the area of the maximum triangle which can be inscribed in a semi-circle of radius 20 inches with its vertex at the centre and its other two angular points on the arc. (D. U.)

3. The strength of a rectangular beam is proportional to the product of its width by square of its depth. Find the strongest beam of rectangular section that can be cut from a log, 24 inches in diameter.

4. The stiffness of a rectangular beam varies as the breadth and the cube of the depth. Show that the stiffest beam which can be cut from a circular log of radius a has depth $\sqrt{3}$ times the breadth.

5. (*i*) Show that the square has the greatest area among rhombi of given side.

(*ii*) Show that the square has the shortest side among rhombi of given area.

6. Find the rhombus of least (*i*) perimeter and (*ii*) area that can be circumscribed to a circle of radius a. (P. U. 1944)

7. Find the rectangle of greatest area whose sides pass through the angular points of a given rectangle of sides a and b.

8. Find the area of the largest rectangle that can be cut from a circular quadrant of radius a.

9. A sphere is cut in the shape of a circular cone. How much of the material can be saved ?

[**Hint**. Find the cone of max. volume inscribed in the sphere].

10. A cone is inscribed in a sphere of radius r, prove that the volume as well as its curved surface is greatest when its altitude is $\frac{4}{3}r$.

11. A given quantity of metal is to be cast into a half cylinder *i.e.* with a rectangular base and semi-circular ends. In order that the total surface area may be a minimium, show that the ratio of the length of the cylinder to the diameter of its semi-circular end is $\pi/(\pi+2)$.

12. A cylinder is inscribed in a right circular cone, prove that (*i*) its curved surface is a maximum when its altitude is $\frac{1}{2}$ that of the cone ; (*ii*) its volume is a maximum when its altitude is $\frac{1}{3}$ that of the cone and show that the volume of this cylinder is $\frac{4}{9}$th that of the cone.

13. From the vertex A of an isosceles triangle ABC, AD is drawn perpendicular to the base. Find the point P on AD such that $PA+PB+PC$ is minimum. (P.U.)

14. Given a fixed straigh line and two points A, B outside it. Find the point P on the line such that (*i*) $AP+BP$ (*ii*) AP^2+BP^2 shall be minimum.

15. A point in the hypotenuse of a right-angled triangle is distant a, and b from the sides. Show that the length of the hypotenuse is at least $(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{3}{2}}$.

(P. U. 1940)

16. A straight line is drawn through the point (a,b) so as to form the triangle of least area with the axes. Find the intercepts it makes on the axes, the axes being rectangular.

17. A wall 10 ft. high is 8ft. from a house. Find the length of the shortest ladder that will reach the house when one end rests on the ground outside the wall.

18. A long ladder has to be moved horizontally through a lane of width d feet containing a right-angled bend. Show that the longest ladder which can be taken round the bend is $2\sqrt{2}\,d$ ft. long.

19. A pole 27ft. long is carried horizontally along a corridor 8ft. wide and into a second corridor at right angles to the first. How wide must the second corridor be ?

20. Prove that the least perimeter of an isosceles triangle which can be circumscribed about a circle of radius a is $6\sqrt{3}a$. (P. U. 1953).

7.73. A few problems of a general character are given below.

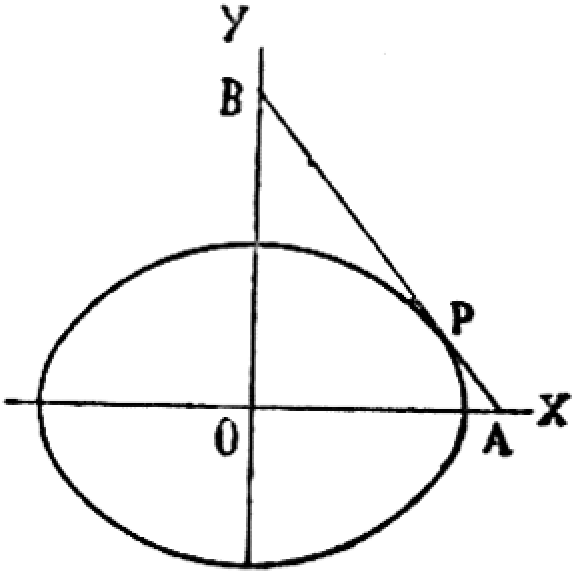
Ex. 1. *Show that the least value of the portion of the tangent to the ellipse $x^2/a^2+y^2/b^2=1$ intercepted between the axes is $a+b$* (P. U. 1950)

$P(a\cos\theta, b\sin\theta)$ is any point on the given ellipse.

Equation of the tagent at this point is $x\cos\theta/a+y\sin\theta/b=1$.

Its intercepts on the co-ordinate axes are

$OA=a\sec\theta$, $OB=b\,\text{cosec}\,\theta$.



Let y denote AB^2 *i. e.* the the square of the length of the tangent at P intercepted between the axes, then

$$y=a^2\sec^2\theta+b^2\text{cosec}^2\theta.$$
$$y_1=2(a^2\sec^2\theta\tan\theta-b^2\text{cosec}^2\theta\cot\theta)$$

This is zero when $\tan^4\theta = b^2/a^2$ or $\tan^2\theta = b/a$
[Sec Ex. 2. Art. 7·5].

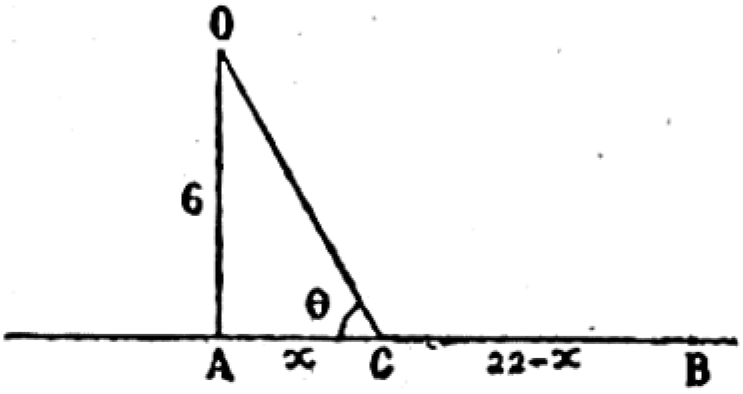
$\therefore$ the least value of $y = a^2\left(1+\frac{b}{a}\right) + b^2\left(1+\frac{a}{b}\right) = (a+b)^3$.

Hence the least value of AB$=a+b$.

Note. The nature of the problem is such that the test of sign of the second derivative, which is not easy in this case, is unnecessary.

Ex. 2. *A man is 6 miles from A, the nearest point to him on a straight road. He wishes to reach a place B on the road 22 miles from A. If he can walk at the rate of 5 miles an hour and can ride a cycle on the road at the rate of 13 miles an hour, where should he arrive on the road so as to reach B in the shortest possible time ?*

Also find the shortest time taken to reach B.



Let O be the original position of the man and AB the straight road, so that OA$=6$ miles and AB$=22$ miles.

Let C be the point on the road where the man ceases to walk and begins to ride a cycle, so that he walks the distance OC and rides the cycle for the distance BC.

Let $\angle$OCA$=\theta$, then AC$=6\cot\theta$ and OC$=6\operatorname{cosec}\theta$, and BC$=22-x=22-6\cot\theta$.

Let T be the total time taken by the man, we have

$$T = \frac{OC}{5} + \frac{BC}{13} = \frac{6\operatorname{cosec}\theta}{5} + \frac{22-6\cot\theta}{13}$$

$$\therefore \frac{dT}{d\theta} = -\tfrac{6}{5}\operatorname{cosec}\theta\cot\theta + \tfrac{6}{13}\operatorname{cosec}^2\theta$$

This is zero when $\frac{1}{5}\cot\theta = \frac{1}{13}$ cesec θ or $\cos\theta = \frac{5}{13}$.

$\therefore$ $AC=6\cot\theta=6\times\frac{5}{12}=\frac{5}{2}=2\frac{1}{2}$ mi.

and $OC=6\operatorname{cosec}\theta=6\times\frac{13}{12}=\frac{13}{2}$ mi. and $BC=22-2\frac{1}{2}$
$=\frac{39}{2}$ mi.

$$\therefore\ T=\frac{OC}{5}+\frac{BC}{13}=\frac{13}{10}+\frac{39}{2\times13}=\frac{13}{10}+1\frac{1}{2}=2\frac{4}{5}\text{hrs.}$$

Aliter. It can also be solved by expressing T in terms of x.

$$\text{Thus}\ \ T=\frac{OC}{5}+\frac{CB}{13}=\frac{\sqrt{36+x^2}}{5}+\frac{22-x}{13}.$$

$$\frac{dT}{dx}=\frac{x}{5\sqrt{36+x^2}}-\frac{1}{13}$$

this is zero when $13x=5\sqrt{36+x^2}$ or $144x^2=25\times36$,
$\therefore$ $x=\frac{5}{2}$ and so on as before.

Exercise XXV

1. A rectangle is drawn with its vertices on the ellipse $x^2/a^2+y^2/b^2=1$ and its sides parallel to the axes.

Find the ratio of the sides when (*i*) the area is max. (*ii*) the perimeter is max.

2. Find the area of the greatest isosceles triangle that can be inscribed in the ellipse given in Q. 1, the triangle having its vertex coincident with an extremity of the major axis.

3. AA′ is the major axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and PQ is any chord parallel to AA′. Prove that the maximum area of the trapezium APQA′ is $\frac{3\sqrt{3}}{4}ab$.

4. Prove that the area of the triangle formed by the tangent at any point on the ellipse given in Q. 1 and its axes is a mininum for the point $(a/\sqrt{2},\ b/\sqrt{2})$.

5. The normal at a point P on the ellipse given in Q. 1. intersects the major axis in G and N is the foot of the ordinate of P. Show that the maximum area of the triangle PGN is $\frac{b^3}{4a}$.

Show also that the max. distance of the normal from the centre of the ellipse is $a-b$.

6. The point $P(9a, -6a)$ on the parabola $y^2=4ax$ is joined to any point $Q(at^2, 2at)$ on the parabola. Show that PQ has two minimum volues 0 and $5\sqrt{5}a$ and a maximum value $8\sqrt{2}a$.

7. From a point on the axis of the parabola $y^2=4ax$ radii vectors are drawn to the curve. If the point be at a distance d from the vertex, find the abscissa of the extermity of the least radius vector.

8. Find at what point the sub-tangent to the curve $xy^2=a^2(a-x)$ is a minimum.

9. If a stone is projected vertically upwards with a velocity of 80 ft. per sec., its hight, in feet, after t seconds is $80t-16t^2$. When will the stone be at the greatest height and what height does it attain ?

10. A ship B is 75 miles due east of a shipA. If B sails west 12 miles per hour and A south 9 miles per hour; findt when the ships will be nearest together.

11. A man in a boat at one bank of a river 1200 ft. wide, wants to go to a point 3600 ft. down the river on the other bank in the minimum time. If he can row 3 miles per hour and can walk 5 miles per hour, how much near the point should he land ?

12. A man in a rowboat 6 miles from the shore desires to reach a point on the shore at a distance of 10 miles from his present position. If he can walk 4 miles per hour and row 2 miles per hour, in what direction should he row in order to reach his destination in the shortest possible time ?

13. A man is 5 miles from A, the nearst point to him of a straight railway ; he wants to reach a station B, 8 miles from A ; if he can walk across country at the rate of $2\frac{1}{2}$ miles an hour and along the line at the rate of 3 miles an hour, where should he join the line to arrive at B in the shortest possible time ?

14. A man is walking along a straight road BCX and wants to get to A as quickly as possible; A lies off the road across rough ground over which his pace is half what it is along the road. The angle BCA being 120°, prove that he must leave the road at C. (D.U. 1926)

15. A man is at A on the edge of a circular lake of diameter d and he wishes to reach the diametrically opposite point B. He can row at 3 m.p.h. and run at 6 m.p.h. He rows in a direction making an angle θ with AB and then runs round the edge of the lake to B. Find the value of θ for which the time is a maximum. Show that to reach B in the least time he should run all the way.

Revision Questions III

1. Prove that if $f'(a)$ vanishes, $f(a)$ is a max. or a min. value of $f(x)$, according as $f''(a)$ is negative or positive. (P.U. 1948)

2. Find the max. and min. values of $x+\frac{13}{x}+\frac{6}{x^2}$.

3. A peasant has 20 hurdles of equal lengths and he wishes to enclose a rectangular cow-shed which shall have a maximum area ; find the number of hurdles he must place on each side of the plot. (D.U. 1932)

4. CAB is an isosceles triangle right-angled at C. From a point P on AB perpendiculars PM, PN are drawn to CA, CB respectively. Find the position of P in which the area of the rectangle PMCN is the greatest.

5. P and Q are two fires 45 ft. apart ; the fire at P gives 8 times as much heat as the fire at Q. If the intensity of heat varies inversely as the square of the distance, find what point on PQ receives the least heat.

6. Assuming that the intenisty of light is inversely proportional to the square of the distance from the source, find the point on the line joining two sources A and B,. A being twice as intense as B at which the illumination is a minimum.

7. The lengths of the parallel sides of a trapezium are a and $a+2x$ and the other sides are equal and of length b. Find the value of x for max. area of the trapezium. (P.U.)

8. At what distance from the wall of a house should a man whose eye is 5 ft. from the ground station himself so that a window 5 ft high whose sill is 20 ft. from the ground may subtend the greatest angle ?

9. A sheet of paper is to contain 18 square inches of printed matter. The margins at the top and bottom are to be 2 inches each, and at the sides 1 inch each.

Find the the dimensions of the sheet which require the least amount of paper. (P.U.)

10. The perimeter of a triangle is 8 inches. If one side is 3 inches, what are the other two for max. area of the tringle ? (P.U.)

11. Prove that of all triangles having the same perimeter the equilateral triangle has the greatest area.

12. Show that the triangle of maximum area inscribed in a circle of radius a is an equilateral triangle. (P.U. 1952)

13. A and B are points on the same side of a plane mirror. A ray of light passes from A to B by way of a point C on the mirror. Show that the length of the path will be least when the lines AC, CB make equal angles with the mirror.

CHAPTER VIII

DIFFERENTIATION OF LOGARITHMIC and EXPONENTIAL FUNCTIONS

8·1. In the process of finding the derivative of $\log x$ we find it necessary to determine the very important limit.

$$e=\underset{n\to\infty}{\mathrm{Lt}}\left(1+\frac{1}{n}\right)^{n}=2{\cdot}7182 \text{ approximately*}.$$

That this limit exists and has the value indicated is proved in more advanced texts. We roughly indicate below that in case n is a positive integer (although the result holds true from all real values of n), the limit lies between 2 and 3.

8·2. $\underset{n\to\infty}{\textbf{Limit}}\left(1+\frac{1}{n}\right)^{n}$.

So long as n is finite, by Binomial theorem, we have

$$\left(1+\frac{1}{n}\right)^{n}=1+n.\frac{1}{n}+\frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^{2}+\frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^{3}+\ldots$$

$$=1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\ldots\ldots(\mathrm{X})$$

As $n\to\infty$ we can suppose n to be a large positive integer so that $\left(1-\frac{1}{n},\right)\left(1-\frac{2}{n}\right)$ are all positive and less than 1.

---

*The student might argue that

$$\underset{n\to\infty}{\mathrm{Lt}}\left(1+\frac{1}{n}\right)^{n}=\left(1+\frac{1}{\infty}\right)^{\infty}=(1+0)^{\infty}=1^{\infty}=1$$

and might think that this limit is equal to 1, but this is incorrect as infinity cannot be treated as an ordinary number to which laws of Algebra are applicable. This limit is in fact an *irrational number* lying between 2 and 3.

$\therefore\ \left(1+\frac{1}{n}\right)^n$ *is evidently greater than 2.* ...(1)

Also $\frac{1}{2!}\left(1-\frac{1}{n}\right)=\frac{1}{2}\times$ a proper fraction $\therefore\ <\frac{1}{2}$,

$\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)=\frac{1}{2.3}\times$ a proper fraction

and $\therefore\ <\frac{1}{2\cdot3}<\frac{1}{2\cdot2}$ or $<\frac{1}{2^2}$.

$\frac{1}{4!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)=\frac{1}{2.3.4}\times$ a proper fraction

$\therefore\ <\frac{1}{2.3.4}<\frac{1}{2.2.2}$ or $<\frac{1}{2^3}$.

$$\therefore\ \left(1+\frac{1}{n}\right)^n<1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\ldots.+\frac{1}{2^{n-1}}$$

$$<1+\left(1+\tfrac{1}{2}+\frac{1}{2^2}+\ldots\ldots.+\frac{1}{2^{n-1}}\right)$$

$$<1+\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}$$

$$<1+2\left(1-\frac{1}{2^n}\right)$$

$$<1+2-\frac{1}{2^{n-1}}<3 \qquad \ldots\ldots(2)$$

Hence $\left(1+\frac{1}{n}\right)^n$ tends to a finite limit lying between 2 and 3 as n tends to infinity through positive integral values of n. This limit is usually denoted by e.

8·21. Writing $y=\left(1+\frac{1}{n}\right)^n$ and computing y when

$n=1, 2, 3, 10, \ldots 100$ etc, we obtain the following table.

n	1	2	3
y	$(1+1)^1=2$	$(1+\frac{1}{2})^2=2{\cdot}25$	$(1+\frac{1}{3})^3=2{\cdot}37037$

n	10	100	1000
y	$(1{\cdot}1)^{10}=2{\cdot}59374$	$(1{\cdot}01)^{100}=2{\cdot}70481$	$(1{\cdot}001)^{1000}=2{\cdot}71692$

This shows that y approaches a limit a little greater than 2·71692.

This can also be seen as follows :—

When $n \to \infty$, $\dfrac{1}{n} \to 0$, and hence from equation(X) we have,

$$e = \underset{n\to\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)^n = 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\ldots$$

$$\begin{aligned}
&=2\\
&+0{\cdot}5\\
&+0{\cdot}166667\\
&+0{\cdot}041667\\
&+0{\cdot}008333\\
&+0{\cdot}001389\\
&+0{\cdot}000198\\
&+0{\cdot}000025\\
&+0{\cdot}000003\\
&\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

$=2{\cdot}718282$ current to five places of decimals.

Note—e is an incommensurable number and resembles π in being an interminable non-recurrent decimal.

Examples

Ex. 1. $\underset{x\to 0}{\text{Lt}}\ (1+x)^{\frac{1}{x}} = e.$

Put $x = \dfrac{1}{n}$, then as $x \to 0$, $n \to \infty$

$$\therefore \quad \underset{x\to 0}{\text{Lt}}\,(1+x)^{\frac{1}{x}}=\underset{n\to\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)^{n}=e.$$

Ex. 2. $$\underset{x\to 0}{\text{Lt}}\,(1+x)^{\frac{a}{x}}=\underset{x\to 0}{\text{Lt}}\left[(1+x)^{\frac{1}{x}}\right]^{a}=e^{a}.$$

Ex. 3. $$\underset{z\to\infty}{\text{Lt}}\left(1+\frac{b}{z}\right)^{z}=\underset{z\to\infty}{\text{Lt}}\left[\left(1+\frac{b}{z}\right)^{\frac{z}{b}}\right]^{b}=e^{b}$$

8·3. To prove that the limit $\frac{a^h-1}{h}$ when $h\to 0$ is $\log_e a$.

Let $a^h-1=x$, then $a^h=1+x$ and $a=\log_a(1+x)$.

$$\therefore \quad \frac{a^h-1}{h}=\frac{x}{\log_a(1+x)}=\frac{1}{\frac{1}{x}\log_a(1+x)}=\frac{1}{\log_a(1+x)^{\frac{1}{x}}}$$

Now when $h\to 0$, $x\to 0$ and $\frac{1}{x}\to\infty$

$$\therefore \quad \underset{h\to 0}{\text{Lt}}\,\frac{a^h-1}{h}=\underset{x\to 0}{\text{Lt}}\,\frac{1}{\log_a(1+x)^{\frac{1}{x}}}=\frac{1}{\log_a e}=\log_e a.$$

Cor. Putting e for a, $\underset{h\to 0}{\text{Lt}}\,\frac{e^h-1}{h}=\log_e e=1$

8·31. In ordinary numerical calculations by the help of logarithms, we use what is called the common system of logarithms in which the number 10 is taken as the base. But in Calculus and in higher Mathematics the number $e=\underset{n\to\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)^{n}$ is taken as the base and this system is called the Natural System of Logarithms.

8·32. The elementary properties of logarithms with which the student is already familiar are :—

$\log_a x+\log_a y=\log_a xy.$ (1) $\log_a x-\log_a y=\log_a\frac{x}{y}.$ (2)

$\log_a x^n = n\log_a x$. (3) $\log_a x = \log_e x \times \log_a e$. (4)

Any quantity $x = a^{\log_a x}$ or $e^{\log_e x}$. (5)

when there is no mention of any base of a logarithm the base e will always be understood.

8·4. Differentiation of $\log_a x$ Let $y = \log_a x$.

When x is increased to $x+\delta x$, let y become $y+\delta y$, so that

$$y+\delta y = \log_a(x+\delta x)$$

$$\therefore \quad \delta y = \log_a x+\delta x) - \log_a x$$

$$= \log_a \frac{x+\delta x}{x} = \log_a\left(1+\frac{\delta x}{x}\right)$$

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{1}{\delta x}\log_a\left(1+\frac{\delta x}{x}\right)$$

$$= \frac{1}{x}\,\frac{x}{\delta x}\log_a\left(1+\frac{\delta x}{x}\right)$$

$$= \frac{1}{x}\log_a\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}}$$

Now as $\delta x \to 0, \frac{x}{\delta x} \to \infty$, $\therefore \underset{\delta x\to 0}{\text{Lt}}\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}} = e$

Hence $\frac{dy}{dx} = \underset{\delta x\to 0}{\text{Lt}}\frac{1}{x}\log_a\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}}$

$$= \frac{1}{x}\log_a e = \frac{1}{x\log_e a}.$$

Cor. (*i*) If $a = e$, $y = \log_e x$ and $\frac{dy}{dx} = \frac{1}{x}\log_e e = \frac{1}{x}$.

i. e., $\quad \mathbf{\frac{d}{dx}(\log x) = \frac{1}{x}}$.

Similarly $\frac{d}{du}(\log u) = \frac{1}{u}$.

But $\quad \frac{d}{dx}(\log u) = \frac{d}{du}(\log u) \times \frac{du}{dx}$

$$= \frac{1}{u}\cdot\frac{du}{dx}.$$

Cor. (*ii*) If $y = \log_a u$ then $\frac{dy}{dx} = \frac{1}{u}\log_a e.\frac{du}{dx}$.

8·5. Differentiation of a^x.

Let $\qquad y=a^x$

then $\qquad \log_e y=\log_e a^x=x \log_e a.$

Differentiating both sides with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx}=\log_e a$$

i. e.,

$$\frac{dy}{dx}=y \log_e a=a^x\log_e a.$$

Cor. (*i*) If $a=e$, then $y=e$ and $\dfrac{dy}{dx}=e^x \log_e e=e^x$.

which shows that *the exponential function* e^x *merely repeats itself by differentiation, a property not possessed by any other function of* x.

(*ii*) If $y=a^u$, $\dfrac{dy}{dx}=\dfrac{d}{du}(a^u)\times\dfrac{du}{dx}=a^u \log_e a.\dfrac{du}{dx}$.

8·5-1. Differentiation of a^x from first principles.

Let $\qquad y=a^x$

then $y+\delta y=a^{x+\delta x}$

$$\delta y=a^{x+\delta x}-a^x=a^x(a^{\delta x}-1).$$

$$\frac{\delta y}{\delta x}=a^x.\frac{a^{\delta x}-1}{\delta x}.$$

$$\therefore \quad \frac{dy}{dx}=a^x.\underset{\delta x\to 0}{\text{Lt}}\frac{a^{\delta x}-1}{\delta x}$$

*[Now let $a^{\delta x}-1=z$, (compare Art. 8·3)

then $z\to 0$ as $\delta x\to 0$, also $\delta x=\log_a(1+z)$

And $\qquad \dfrac{a^{\delta x}-1}{\delta x}=\dfrac{z}{\log_a(1+z)}=\dfrac{1}{\frac{1}{z}\log_a(1+z)}=\dfrac{1}{\log_a(1+z)^{\frac{1}{z}}}$

$$\therefore \quad \underset{\delta x\to 0}{\text{Lt}} \frac{a^{\delta x}-1}{\delta x}=\underset{z\to 0}{\text{Lt}} \frac{1}{\log_a(1+z)^{\frac{1}{x}}}=\frac{1}{\log_a e}=\log_e a]$$

Hence $$\frac{dy}{dx}=a^x \log_e a.$$

Ex. Deduce the differential co-efficient of $\log_a x$ from that of a^x. [Let $y=\log_a x$, then $x=a^y$, find dx/dy].

Examples

Ex. 1. If $y=\log \sqrt[5]{x^3}=\log x^{\frac{3}{5}}=\frac{3}{5}\log x$, then $\frac{dy}{dx}=\frac{3}{5x}$.

Ex. 2. If $y=a^{mx}$, then $\frac{dy}{dx}=a^{mx} \log a\times \frac{d}{dx}(mx)$

$$=ma^{mx}\log a.$$

Ex. 3. If $y=\epsilon^{nz}$, then $\frac{dy}{dx}=\frac{d}{dz}(\epsilon^{nz}).\frac{dz}{dx}=n\epsilon^{nz}\frac{dz}{dx}$.

Ex. 4. If $y=e^{\tan x}\ \frac{dy}{dx}=e^{\tan x}\frac{d}{dx}(\tan x)=e^{\tan x}.\sec^2 x$.

Ex. 5. If $y=\log(\sin^{-1}x)$, find $\frac{dy}{dx}$.

Let $z=\sin^{-1}x$, then $\frac{dz}{dx}=\frac{1}{\sqrt{1-x^2}}$ and $y=\log z$

$$\therefore \quad \frac{dy}{dx}=\frac{dy}{dz}\times\frac{dz}{dx}=\frac{1}{z}.\frac{1}{\sqrt{1-x^2}}=\frac{1}{\sqrt{1-x^2}\sin^{-1}x}.$$

Ex. 6. Differentiate $y=\epsilon^{ax}\log\sin bx$.

$$\frac{dy}{dx}=\frac{d}{dx}(\epsilon^{ax})\times\log\sin bx+\epsilon^{ax}\frac{d}{dx}(\log\sin bx)$$

$$=a\epsilon^{ax}\log\sin bx+\epsilon^{ax}\frac{1}{\sin bx}\times\cos bx\times(b)$$

$$=a\epsilon^{ax}\log\sin bx+b\epsilon^{ax}\cot bx$$

Ex. 7. Differentiate $\log\left[\epsilon^x\left(\frac{x-2}{x+2}\right)\right]$.

(P. U. 1949)

Using the laws of logarithms, we have

$$y=\log e^x+\tfrac{3}{4}\log\frac{x-2}{x+2}=x+\tfrac{3}{4}\log(x-2)-\tfrac{3}{4}\log(x+2),$$

$$\therefore\ \frac{dy}{dx}=1+\tfrac{3}{4}\left(\frac{1}{x-2}-\frac{1}{x+2}\right)=1+\tfrac{3}{4}.\ \frac{(x+2)-(x-2)}{x^2-4}$$

$$=1+\frac{3}{x^2-4}=\frac{x^2-1}{x^2-4}.$$

Ex. 8. Differentiate log $\tan^{-1}x$ with respect to $\cot^{-1}x$.

Let $y=\log\tan^{-1}x$,

then $$\frac{dy}{dx}=\frac{1}{\tan^{-1}x}\times\frac{d}{dx}(\tan^{-1}x)=\frac{1}{(1+x^2)\tan^{-1}x}$$

and let $u=\cot^{-1}x$; then $$\frac{du}{dx}=-\frac{1}{1+x^2}$$

$$\text{or}\ \frac{dx}{du}=-(1+x^2)$$

$$\therefore\ \frac{dy}{du}=\frac{dy}{dx}\times\frac{dx}{du}=\frac{1}{(1+x^2)\tan^{-1}x}\times[-(1+x^2)]$$

$$=-\frac{1}{\tan^{-1}x}.$$

8·6. Formulae for differentiation. The formulae derived so far in this chapter are collected below for reference :

1. $\displaystyle \mathop{\mathrm{Lt}}_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e.$

2. $\displaystyle \mathop{\mathrm{Lt}}_{h\to0}\frac{a^h-1}{h}=\log_e a;\qquad \mathop{\mathrm{Lt}}_{h\to0}\frac{e^h-1}{h}=1.$

3. $\displaystyle \frac{d}{dx}(\log_a x)=\frac{1}{x\log_e a}.$ 4. $\displaystyle \frac{d}{dx}(\log_e u)=\frac{1}{u}.\frac{du}{dx}.$

5. $\displaystyle \frac{d}{dx}(a^x)=a^x\log_e a.$ 6. $\displaystyle \frac{d}{dx}(e^x)=e^x.$

E[illegible] XXVI-A.

Find the derivatives of :—

1. (i) $\log(7-5x)$. (ii) $\log(3x^2+5x+1)$

✓(iii) $\log_{10}(x^3-3x)^{\frac{1}{3}}$ (iv) $\log(\sec x+\tan x)$

(v) $t^2\log t$. (vi) $\sin x\log\sin x$.

✓**2.** (i) $e^{-\frac{1}{2}x^2}$ (ii) 10^{-x^2}. (iii) $e^{\cot\theta}$

(iv) $x^n.e^{-x}$. (v) $x^n.n^x$. (vi) x^e+e^x

(vii) $x^e.e^x$. (viii) $(3x+1)e^{-3x}$

3. (i) $\log\sqrt{\frac{x+a}{x-a}}$. (ii) $\log\sqrt{x^2-a^2}$.

✓(iii) $\tan^{-1}x+\log\sqrt{\frac{x-1}{x+1}}$. (iv) $\log(\log x)$.

(v) $\log(x+\sqrt{1+x^2})$. (vi) $\log\frac{x^2+x+1}{x^2-x+1}$.

✓(vii) $\frac{e^x-e^{-x}}{e^x+e^{-x}}$ ✓(viii) $x\sec^{-1}x-\log(x+\sqrt{x^2-1})$.

4. Find the values of the derivatives of the following for $x=2$:

(i) $\log\sqrt{x}$. (ii) $\log\frac{1}{\sqrt{x}}$. ✓(iii) $\sqrt{\log x}$.

(iv) $\sqrt{\log\sqrt{x}}$. (v) $\tan^{-1}(\log x)$.

5. Differentiate :—

✓(i) $\frac{1}{2}x[\sin\log(x)-\cos(\log x)]$. (ii) $\log\sin x$.

(iii) $\log\sin x^2$. (iv) $\log(1+\sin\theta.)$.

(v) $\log\frac{1+\sin\theta}{1-\sin\theta}$. (vi) $\log\sqrt{\frac{1+\sin x}{1-\sin x}}$.

6. (i) $\log\frac{1-x\sin x}{1+x\sin x}$. (ii) $\frac{\log x}{1+x\log x}$.

(iii) $\frac{\log\sin x}{\log\cos x}$.

7. $x^5 \log a + 5a^x + 2\tan^{-1}bx - \log(1-xb) + \log(1+xb)$. (D. U. 1932)

8. (*i*) $\tan^{-1}(ac^x x^2)$. (P. U. 1944)

(*ii*) $2^x e^{\sin^{-1}x}$. (P. U. 1950)

(*iii*) $\log \dfrac{a+b\sin x}{b-a\sin x}$ (P.U. 1953).

9. Differentiate (*i*) a^x with regard to $\log x$.

(*ii*) $e^x/(1+e^x)$ with regard to e^x.

10. (*i*) If $y = e^{3\log x}$, show that $\dfrac{dy}{dx} = 3x^2$. (P. U. 1949)

(*ii*) If $y \log x = x - y$, prove that $\dfrac{dy}{dx} = \dfrac{\log x}{(1+\log x)^2}$. (P. U. 1948)

(*iii*) If $r = ae^{\cot\theta}$, show that $\dfrac{dr}{d\theta} + r\,\mathrm{cosec}^2\theta = 0$.

(*iv*) If $x = e^{-2t}\cos 2t$, show that $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 8x = 0$

(*v*) If $y = \log(x + \sqrt{1+x^2})$, ~~prove~~ that

$$(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0. \qquad \text{(P.U. 1948)}$$

(*vi*) If $y = \log\left(\dfrac{x}{a+bx}\right)^x$, prove that

$$x^3\frac{d^2y}{dx^2} = \left(y - x\frac{dy}{dx}\right)^2. \qquad \text{(P. U. 1944)}$$

11. Determine the values of A and B so that

$$\frac{d}{dx}\left[e^{-2x}(\mathrm{A}\sin x + \mathrm{B}\cos x)\right]$$

may be identically equal $e^{-2x}\cos x$.

12. If $u = e^x + e^{-x}$ and $v = e^x - e^{-x}$, prove that

(*i*) $\dfrac{du}{dx} = v$. (*ii*) $\dfrac{dv}{dx} = u$. (*iii*) $\dfrac{d}{dx}(uv) = u^2 + v^2$.

Exercise XXVI-B

1. Find the equation of the tangent to the curve $y=\log x$ at the point for which $x=a$ and find a so that the tangent may pass through the origin.

2. Prove that $\frac{x}{a}+\frac{y}{b}=1$ touches the curve

$$y=be^{-\frac{x}{a}}$$

at the point where the curve crosses the axis of y.

(P. U. 1951).

3. Find the equation of the common tangent to $y=\log x$ and $y=x\log x$ at their common point.

4. Show that the sub-tangent is of constant length for the curve $y=a^x$.

5. Show that in the exponential curve $y=be^{\frac{x}{a}}$ the sub-tangent is of constant length and the sub-normal varies as the square of the ordinate. (P. U. 1951).

6. Show that in the curve $y=a\log\sec\frac{x}{a}$ the abscissa at any point varies as the angle which the tangent line makes with the axis of x.

7. Prove that the length of the normal for the

chainette $$y=\frac{c}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right)$$

varies as the square of the ordinate of the point.

8. Show that in the curve $y=a\log(x^2-a^2)$, the sum of the lengths of the tangent and the sub-tangent vary as the product of the co-ordinates of the point.

9. Find the maximum and minimum values of,

(i) xe^x, (ii) $\frac{x}{\log x}$. (iii) $e^x\sin x$.

10. Find the least value of $ae^{cx}+be^{-cx}$.

11. Prove that $x^{\frac{1}{x}}$ has a maximum value for $x=e$. (P. U. 1943).

12. Find the maximum value of $\frac{1}{x}\log x$. (P.U. 1944)

13. Show that the function $\left(\frac{1}{x}\right)^x$ is maximum when $x=1/e$.

14. If $y=x^2 \log\frac{1}{x}$, show that the maximum value of y is $1/2e$.

15. Show that the maximum and minimum values of $(1-x)^2 e^x$ are respectively $4/e$ and zero.

8·7 Derivative of a variable with variable exponent.

Let $y = u^v \quad (u>o)$

where both u and v are functions of x.

Taking logarithm of both sides, we have

$$\log y=\log u^v = v \log u$$

Differentiating with respect to x, we have

$$\frac{1}{y}\frac{dy}{dx}=\frac{d}{dx}[v \log u]= \log u\frac{dv}{dx} + v\frac{1}{u}\cdot\frac{du}{dx}$$

$$\therefore \quad \frac{dy}{dx} = y\left[\log u\frac{dv}{dx}+\frac{v}{u}\frac{du}{dx}\right]=u^v \log u\frac{dv}{dx}+vu^{v-1}\frac{du}{dx}$$

Note. In examples of this kind, it is simpler not to use the general formula obtained above but to take the logarithm of both members before differentiating.

Examples

Ex. 1. *Differentiate* x^x.

Let $\quad y=x^x$.

Then $\log y=x \log x$.

$$\therefore \quad \frac{1}{y}\frac{dy}{dx}=\frac{d}{dx}[\, x \log x\,]=1.\ \log x+x.\frac{1}{x}=\log x+1$$

$$\therefore \quad \frac{dy}{dx}=y\ [\ \log x+1\ ]=y \log (ex)=x^x \log (ex).$$

Ex. 2. If $y = x^{\sin x}$, find $\dfrac{dy}{dx}$.

Here $\log y = \sin x \log x$.

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = \log x \frac{d}{dx}(\sin x) + \frac{d}{dx}(\log x) \sin x$$

$$= \cos x \log x + \frac{1}{x}\sin x.$$

Hence $\quad \dfrac{dy}{dx} = x^{\sin x}\left[\cos x \log x + \dfrac{1}{x}\sin x\right].$

Ex. 3. Differentiate $x^x + x^{\sin x}$ with respect to x.
(P.U. 1944)

Let $\quad y = x^x + x^{\sin x} = u + v.$

$$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$= x^x \log(ex) + x^{\sin x}\left[\cos x \log x + \frac{\sin x}{x}\right].$$

(From Ex. 1 and Ex. 2)

Note. The student is warned against committing a mistake of the following type in such questions :—

$$y = x^x + x^{\sin x};\ \log y = x \log x + \sin x \log x.$$

In such cases, the derivatives of the two functions should be obtained separately and the results added.

Exercise XXVII—A

Differentiate :—

1. (*i*) x^{x^2}. (*ii*) $(1+x^2)^{2x}$. (*iii*) $x^{\frac{1}{x}}$.

(*iv*) $(x)^{e^x}$. (*v*) $(e^x)^x$. (*vi*) $(\log x)^x$.

(*vii*) $x^{\log x}$. (*viii*) $x^{\tan^{-1} x}$. (*ix*) $(\tan^{-1} x)^x$.

(*x*) $(\sin x)^{\log x}$ (*xi*) $(\log x)^{\sin x}$ (*xii*) $(\cos x)^x$.

(*xiii*) $\left(1 + \dfrac{1}{x}\right)^x$. (*xiv*) $(\cos^{-1} x)^x$.

2. (*i*) $(1+x^2)^{7x}$ (P.U. 1948) (*ii*) $(\cot^2 x)^{\frac{1}{x}}$ (P.U. 1945)

(*iii*) $(\tan x)^{\frac{1}{x}}$ (P.U. 1949)

(*iv*) $x^x + (\cos x)^{\frac{1}{x}}$. (P.U. 1950)

(*v*) $x^{\tan x} + (\tan ax)^{ax}$. (P.U. 1952)

(*vi*) $(\sin x)^{\tan x} + (\tan x)^{\sin x}$. (P.U. 1952S)

3. Find $\dfrac{dy}{dx}$ when $y = x^x + x^{\frac{1}{x}}$. (P.U.)

8·71. The differentiation of certain complicated functions becomes simpler by taking logarithm before differentiating.

Ex. 1. Differentiate

(*i*) $y = \dfrac{x\sqrt{1+x}}{(1+x^2)^{\frac{3}{2}}}$. (*ii*) $z = \sqrt{\dfrac{1-\sec x}{1+\sec x}}\ \operatorname{cosec} x$

(*i*) Using the laws of logarithm, we have,

$$\log y = \log x + \tfrac{1}{2}\log(1+x) - \tfrac{3}{2}\log(1+x^2).$$

Differentiating with respect to x, we have

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2(1+x)} - \frac{3}{2} \times \frac{2x}{1+x^2}$$

$$= \frac{2(1+x)(1+x^2) + x(1+x^2) - 6x^2(1+x)}{2x(1+x)(1+x^2)}$$

$$= \frac{2+3x-4x^2-3x^3}{2x(1+x)(1+x^2)}$$

$$\therefore \frac{dy}{dx} = \frac{x\sqrt{1+x}}{(1+x^2)^{\frac{3}{2}}} \times \frac{2+3x-4x^2-3x^3}{2x(1+x)(1+x^2)}$$

$$= \frac{2+3x-4x^2-3x^3}{2\sqrt{1+x}\,(1+x^2)^{\frac{5}{2}}}.$$

(*ii*) Here $\log z = \log \operatorname{cosec} x + \frac{1}{2}\log(1-\sec x)$
$-\frac{1}{2}\log(1+\sec x)$

$$\therefore \frac{1}{z}\frac{dz}{dx} = \frac{-\operatorname{cosec} x \cot x}{\operatorname{cosec} x} + \frac{-\sec x \tan x}{2(1-\sec x)} - \frac{\sec x \tan x}{2(1+\sec x)}$$

$$= -\cot x - \frac{2\sec x \tan x}{2(1-\sec^2 x)}$$

$$= -\cot x + \frac{\sec x \tan x}{\tan^2 x} = -\frac{\cos x}{\sin x} + \frac{1}{\sin x}$$

$$= \frac{1-\cos x}{\sin x} = \tan\frac{x}{2}$$

$$\therefore \frac{dz}{dx} = z \tan\frac{x}{2}$$

$$= \tan\frac{x}{2} \operatorname{cosec} x \sqrt{(1-\sec x)/(1+\sec x)}.$$

Ex. 2. Find $\frac{dy}{dx}$ when $y = x^{x^{x^{\cdots\cdots \text{to} \infty}}}$

We have,

$$y = x^y,$$

$$\therefore \log y = y \log x$$

Differentiating with respect to x, we have

$$\frac{1}{y}\frac{dy}{dx} = \log x \frac{dy}{dx} + y \cdot \frac{1}{x} \text{ or } \frac{dy}{dx} = y \log x \frac{dy}{dx} + \frac{y^2}{x}$$

$$\therefore (1-y\log x)\frac{dy}{dx} = \frac{y^2}{x}$$

i.e.

$$\frac{dy}{dx} = \frac{y^2}{x(1-y\log x)}.$$

Exercise XXVII-B

Differentiate :—

1. (*i*) $(2x+a)^m(3x-a)^n$. (*ii*) $\frac{\sqrt{x+2}}{\sqrt{2-x}}$.

(*iii*) $\frac{(x+2)^3}{(x+1)^4(x+4)^2}$. (*iv*) $\frac{x^2(5+x)^3}{(3+x)^5}$.

(*v*) $\dfrac{(x+1)^{\frac{3}{2}}}{(x-2)^{\frac{7}{2}}(x+3)^{\frac{5}{3}}}$. (*vi*) $\dfrac{\sqrt{x}(x^2+2)^3}{x+1}$.

(P.U. 1945)

2. (*i*) $\dfrac{\sin^m x}{\cos^n x}$. (*ii*) $\sqrt{\dfrac{1-\cos x}{1+\cos x}}$.

(*iii*) $\sqrt{\dfrac{1+\tan x}{1-\tan x}}$ (*iv*) $2^x e^{\sin^{-1}x}$. (P.U. 1950)

(*v*) $a^{2x}.e^{\sqrt{x^2+1}}$. (P.U. 1951)

3. (*i*) $2^x \cot x/\sqrt{x}$. (*ii*) $x^x \sin x^0$. (P.U. 1947)

4. Find $\dfrac{dy}{dx}$ when (*i*) $y=x+\cfrac{1}{x+\cfrac{1}{x+\ldots \text{ to } \infty}}$

(*ii*) $y=\sqrt{x+\sqrt{x+\sqrt{x+\ldots \text{to} \infty}}}$

(*iii*) $y=(\cos x)^{(\cos x)^{(\cos x)\ldots\text{to } \infty}}$.

CHAPTER IX

SUCCESSIVE DIFFERENTIATION

Derivative of the derivative is called a second derivative, the derivative of the *second derivative* is called the *third derivative* and so on. The first and higher dorivatives of $y=f(x)$ are denoted by $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\ldots, \frac{d^ny}{dx^n}$, the last being a derivative of the nth order. They are denoted also by $y_1, y_2, y_3, \ldots y_n$ or by $f'(x), f''(x), f'''(x), \ldots\ldots f^n(x)$.

We now find the derivatives of the nth order for some functions of x. The derivative of any other order can be immediately deduced from them by the substitution of the proper value of n.

9·1 *The* n*th derivative of* $(ax+b)^m$ *where* m *is any rational number,*

Let $y=(ax+b)^m$

then
$$y_1=am(ax+b)^{m-1}$$
$$y_2=a^2m(m-1)(ax+b)^{m-2}$$
$$y_3=a^3m(m-1)(m-2)(ax+b)^{m-3}.$$
$$\therefore\ \mathbf{y_n=a^nm(m-1)(m-2)\ldots\ldots(m-n+1)(ax+b)^{m-n}} \tag{1}$$

Cor. (*i*) If $y=x^m$,

then $y_n=m(m-1)(m-2)\ldots\ldots(m-n+1)x^{m-n}$

(*ii*) Further if m is a positive integer and $m=n$, then $y_m=m(m-1)(m-1)\ldots\ldots\ldots\ldots\ldots 2\cdot 1=m!$ (a constant) and therefore, $y_{m+1}=0=y_{m+2}=y_{m+3}\ldots\ldots$

i. e. when m is a positive integer the derivatives of x^m or of $(ax+b)^m$ of the order higher than m, all vanish.

(*iii*) When the index m is a negative integer or a fraction, positive or negative, the nth derivative does not vanish for any value of n, as the power of $ax+b$ or of x does not vanish in that case.

Note. The general result of the equation (1) is written down on the analogy of particular cases but if necessary, a proof by the method of induction, can be given as follows :—

Assume $y_r=a^r m(m-1)(m-2)\ldots(m-r+1)(ax+b)^{m-r}$ (i)
then differentiating this equation, we have
$y_{r+1}=a^{r+1}m(m-1)(m-2)\ldots\ldots(m-r+1)(m-r)(ax+b)^{m-r-1}$ (ii)

This is the same as (i), the integer r being replaced by $r+1$. Thus if the general result (1) is true for the order r, it is also true for the order $r+1$. Hence the result follows. This note will be found applicable to other cases also.

9·2. *The nth derivative of* e^{mx}.

Let $f(x)=e^{mx}$
then $f'(x)=me^{mx}$, $f''(x)=m^2e^{mx}$
$f'''(x)=m^3e^{mx}$ and so on.
Hence $\mathbf{f^n(x)=m^ne^{mx}}$......... (2)

Cor. If $m=1$, $f^n(x)=e^x$ *i. e.* e^x remains unaffected by differentiation.

Note. As indicated in note of Art. 9·1, we can prove result (2) by induction. If $f^r(x)=m^re^{mx}$ then $f^{r+1}(x)=m^{r+1}e^{mx}$, so that if the result is true for $n=r$; it is also true for $n=r+1$.

9·3 *The nth derivative of* a^x.

Let $y=a^x$ then $y_1=a^x\log a$
$y_2=a^x\log a.\log a=a^x(\log a)^2$
$y_3=a^x\log a(\log a)^2=a^x(\log a)^3$ and so on.
Hence $\mathbf{y_n=a^x(\log a)^n}$. ......(2)

Note.—Since any quantity $z=e^{\log z}$ we can write

$$a^x=e^{\log a^x}=e^{x\log a}.$$

Hence $\frac{d^n}{dx^n}(a^x)=(\log a)^n\, e^{x\log a}$ by Art. 9·2

$$=(\log a)^n a^x.$$

9·4. *The nth derivative of* $\sin(ax+b)$.

Let $y=\sin(ax+b)$

then $y_1=a\cos(ax+b)=a\sin\left(ax+b+\frac{\pi}{2}\right)$

$[\because \sin(\tfrac{1}{2}\pi+\theta)=\cos\theta.]$

$y_2=a^2\cos\left(ax+b+\frac{\pi}{2}\right)=a^2\sin\left(ax+b+2.\frac{\pi}{2}\right)$

$y_3=a^3\cos\left(ax+b+2\frac{\pi}{2}\right)=a^3\sin\left(ax+b+3.\frac{\pi}{2}\right)$

and so on.

Hence $\mathbf{y_n=a^n\sin\left(ax+b+n.\frac{\pi}{2}\right)}$ ...(4)

Cor. If $y=\sin ax$, $y_n=a^n\sin\left(ax+n.\frac{\pi}{2}\right)$

$y=\sin 1x$ $y_n=1^n.\sin\left(x+\frac{n\pi}{2}\right)$

and if $y=\sin x$, $y_n=\sin\left(x+n\frac{\pi}{2}\right)$.

9·41. *The nth derivative of* $\cos(ax+b)$.

Let $y=\cos(ax+b)$

then $Dy=-a\sin(ax+b)=a\cos\left(ax+b+\frac{\pi}{2}\right)$

$[\because \cos(\tfrac{1}{2}\pi+\theta)=-\sin\theta.]$

$D^2y=-a^2\sin\left[ax+b+\frac{\pi}{2}\right]=a^2\cos\left[ax+b+2.\frac{\pi}{2}\right]$

and so on.

Hence $D^ny=a^n\cos\left[ax+b+n.\frac{\pi}{2}\right]$.

Cor. If $y=\cos ax$, $D^ny=a^n\cos\left(ax+n.\frac{\pi}{2}\right)$

and if $y=\cos x$, $D^ny=\cos\left(x+n.\frac{\pi}{2}\right)$

9·5. *The nth derivative of* $\dfrac{1}{ax+b}$.

Let $y = \dfrac{1}{ax+b} = (ax+b)^{-1}$

then $y_1 = -a(ax+b)^{-2} = (-1)a(ax+b)^{-2}$,

$y_2 = 1 \cdot 2a^2(ax+b)^{-3} = (-1)^2 2!\, a^2(ax+b)^{-3}$

$y_3 = -1.2.3\, a^3(ax+b)^{-4} = (-1)^3.3!\, a^3(ax+b)^{-4}$

and so on.

$$\therefore\ y_n = (-1)^n n!\, a^n (ax+b)^{-n-1} = \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}} \qquad \ldots(5)$$

Cor. (*i*) If $y = \dfrac{1}{x+b}$, $y_n = \dfrac{(-1)^n n!}{(x+b)^{n+1}}$.

and (*ii*) If $y = \dfrac{1}{x}$, $y_n = \dfrac{(-1)^n n!}{x^{n+1}}$.

a^n

$n a^{n-1}$

9.6. *The nth derivative of log (ax+b).*

Let $y = \log(ax+b)$

then $y_1 = \dfrac{1}{ax+b}.a = a(ax+b)^{-1}$

a^{-n}

$-n a^{-n-1}$

$y_2 = a(-1)(ax+b)^{-2}a = (-1)a^2(ax+b)^{-2}$

$y_3 = (-1)(-2)a^2(ax+b)^{-3}.a = (-1)^2 2!\, a^3(ax+b)^{-3}$

Similarly $y_4 = (-1)^3 3!\, a^4(ax+b)^{-4}$ and so on

Hence $y_n = (-1)^{n-1}(n-1)!\, a^n(ax+b)^{-n}$

$$= \frac{(-1)^{n-1}(n-1)!\, a^n}{(ax+b)^n}$$

Cor. (*i*) If $y = \log(x+b)$, $y_n = \dfrac{(-1)^{n-1}(n-1)!}{(x+b)^n}$

(*ii*) If $y = \log x$, $y_n = \dfrac{(-1)^{n-1}(n-1)!}{x^n}$

Note. Compare this with Art. 9·5.

Examples

Ex. 1. Find the *n*th differential coefficient of $\cos^2 x$. (P.U. 1951)

Let $y = \cos^2 x = \frac{1}{2}.2\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$

$\therefore$ by Art 9.41, $y_n = \frac{1}{2}.\, 2^n \cos\left(2x + n.\dfrac{\pi}{2}\right)$

$$=2^{n-1}\cos\left(2x+\frac{n\pi}{2}\right).$$

Ex. 2. Find the nth differential coefficient of $\sin^3 x$. (P. U. 1949)

Here $y=\sin^3 x=\frac{1}{4}.\ 4\sin^3 x=\frac{1}{4}\ (3\sin x-\sin 3x)$
Hence by Art 9 4,

$$y_n=\tfrac{3}{4}\sin\left(x+\tfrac{1}{2}n\pi\right)-\tfrac{1}{4}.\ 3^n\sin\left(3x+\tfrac{1}{2}n\pi\right)$$

Ex. 3. If $y=\sin 3x\cos 2x$, find y_n. (P.U. 1950)
we have $y=\frac{1}{2}\ (\sin 5x+\sin x)$,

$$\therefore\quad y_n=\tfrac{1}{2}\left[5^n\sin\left(5x+\tfrac{1}{2}n\pi\right)+\sin\left(x+\tfrac{1}{2}n\pi\right)\right].$$

Ex 4. Find the nth differential coefficients of

(*i*) $\dfrac{1}{x^2-a^2}$ (*ii*) $\dfrac{1}{(x+a)^5}$

(*i*) Let $y=\dfrac{1}{x^2-a^2}=\dfrac{1}{(x-a)\ (x+a)}=\dfrac{1}{2a}\left[\dfrac{1}{x-a}-\dfrac{1}{x+a}\right]$

Hence by Art. 9·5, $y_n=\dfrac{1}{2a}\left[\dfrac{(-1)^n n\,!}{(x-a)^{n+1}}-\dfrac{(-1)^n\,n!}{(x+a)^{n+1}}\right]$

(*ii*) Here $y=\dfrac{1}{(x+a)^5}=(x+a)^{-5}$

$\therefore\ y_1=-5\ (x+a)^{-6},\ y_2=-5\ (-6)\ (x+a)^{-7}$

$$=(-1)^2\frac{6\,!}{4\,!}\ (x+a)^{-7}$$

$$y_3=-5.6.7\ \ (x+a)^{-8}=(-1)^3\ \frac{7\,!}{4\,!}(x+a)^{-8}$$

Hence $y_n=(-1)^n\ \dfrac{(n+4)\,!}{4\,!}.\ \dfrac{1}{(x+a)^{n+5}}$

9·7. *The nth differential co-efficient of* $e^{ax}\sin bx$.

Let $y=e^{ax}\sin bx$ .....(1)

$$y_1=a\ e^{ax}\sin bx+be^{ax}\cos bx$$
$$=e^{ax}\ [a\sin bx+b\cos bx]$$

Put $a=k\cos\theta,\ b=k\sin\theta$,

so that $k=\sqrt{a^2+b^2}$, and $\tan\theta=\dfrac{b}{a}$, then we have

$$y_1=\epsilon^{ax}(a^2+b^2)^{\frac{1}{2}}[\cos\theta \sin bx+\sin\theta \cos bx]$$

$$=\epsilon^{ax}(a^2+b^2)^{\frac{1}{2}}\sin(bx+\theta) \qquad \text{...(2)}$$

Comparing (1) and (2) it is evident that y_1 is obtained, from y by increasing the angle bx by θ and by introducing the factor $\sqrt{a^2+b^2}$. By a similar operation on y_1 we have

$$y_2=(a^2+b^2)^{\frac{2}{2}}\epsilon^{ax}\sin(bx+2\theta).$$

Similarly $\quad y_3=(a^2+b^2)^{\frac{3}{2}}\epsilon^{ax}\sin(bx+3\theta),$

and hence $y_n=(a^2+b^2)^{\frac{n}{2}}\epsilon^{ax}\sin(bx+n\theta),$

where θ is given by the equation $\tan\theta=\dfrac{b}{a}$

$$\text{or} \qquad \theta=\tan^{-1}\frac{b}{a}.$$

9·71. *The nth derivative of $\epsilon^{ax}\cos bx$.*

Let $y=\epsilon^{ax}\cos bx \qquad \text{...(1)}$

then $y_1=a\epsilon^{ax}\cos bx-b\epsilon^{ax}\sin bx$

$\qquad =\epsilon^{ax}(a\cos bx-b\sin bx)$

Put $a=k\cos\theta$, $b=k\sin\theta$, so that $k=\sqrt{a^2+b^2}$, $\tan\theta=\dfrac{b}{a}$,

$$\therefore\ y_1=\epsilon^{ax}k(\cos\theta\cos bx-\sin\theta\sin bx)$$

$$=\epsilon^{ax}(a^2+b^2)^{\frac{1}{2}}\cos(bx+\theta) \qquad \text{...(2)}$$

Camparing (1) and (2), it is evident that y_1 is obtained from y by increasing the angle bx by θ and by introducing the factor $(a^2+b^2)^{\frac{1}{2}}$.

By a similar operation on y_1, we have

$$y_2=(a^2+b^2)^{\frac{2}{2}}\epsilon^{ax}\cos(bx+2\theta)$$

similarly $y_3=(a^2+b^2)^{\frac{3}{2}}\, e^{ax}\cos(bx+3\theta)$

Hence $y_n=(a^2+b^2)^{\frac{n}{2}}\, e^{ax}\cos(bx+n\theta)$

where θ is given by $\tan\theta=\dfrac{b}{a}$ i.e. $\theta=\tan^{-1}\dfrac{b}{a}$

9·8. The formulae obtained in this chapter are collected for reference.

1. $y=x^m$, $\quad y_n=m(m-1)(m-2)\ldots(m-n+1)x^{m-n}$
2. $y=e^{mx}$, $\quad y_n=m^n e^{mx}$,
3. $y=a^x$, $\quad y_n=a^x(\log a)^n$.
4. $y=\sin ax$, $\quad y_n=a^n\sin\left[ax+n.\dfrac{\pi}{2}\right]$.
5. $y=\cos ax$, $\quad y_n=a^n\cos\left[ax+n.\dfrac{\pi}{2}\right]$.
6. $y=\dfrac{1}{x+a}$, $\quad y_n=\dfrac{(-1)^n\, n\,!}{(x+a)^{n+1}}$.
7. $y=\log(x+a)$, $y_n=\dfrac{(-1)^{n-1}(n-1)\,!}{(x+a)^n}$.
8. $y=e^{ax}\sin bx$, $y_n=(a^2+b^2)^{\frac{n}{2}}\, e^{ax}\sin(bx+n\theta)$
9. $y=e^{ax}\cos bx$, $y_n=(a^2+b^3)^{\frac{n}{2}}\, e^{ax}\cos(bx+n\theta)$

where $\tan\theta=\dfrac{b}{a}$.

Exercise XXVIII

Find the nth differential co-efficients of :—

1 (*i*) $(ax+b)^{p/q}$ (*ii*) e^{ax+b} A (*iii*) a^{5x}

(*iv*) $e^{2x}-e^{-2x}$ (*v*) $\log(5x+6)$ (*vi*) $\log(1+x)^{\frac{5}{7}}$

(*vii*) $\cos(3x+4)$

2 (*i*) $\sin^2 x$ (P.U. 1948) (*ii*) $\cos^3 x$ (P.U, 1952)

(*iii*) $\sin 2x\sin 3x$ (P U. 1951) (*iv*) 3^{2x} (P.U. 1950)

(*v*) $\sin\sqrt{2x}$ (P.U. 1951)

(*vi*) $\log(ax+x^2)$ (P.U. 1942)

(*vii*) $\dfrac{1}{(x+2)(x-1)}$ (P.U. 1948)

(*viii*) $\dfrac{1}{x^2-(a+b)\,x+ab}$ (P.U. 1941)

(*ix*) $\dfrac{2x+11}{x+5}$ (P.U. 1943)

3 (*i*) $\sin 5x \cos 3x$ (*ii*) $\log(6+5x+x^2)$

(*iii*) $\dfrac{1}{(ax+b)^3}$ (*iv*) $\dfrac{1}{(3-2x)^3}$

4 (*i*) $e^x \sin x$ ✓ (P.U.) (*ii*) $e^x \cos\sqrt{3}x$ (P.U.)

(*iii*) $e^{5x} \sin x$ (P.U.)

5 (*i*) If $y=e^{ax}\sin bx$, show that

$$\frac{d^2y}{dx^2}+(a^2+b^2)\,y=2a\frac{dy}{dx}$$ (P.U. 1953)

(*ii*) If $y=a\cos(\log x)+b\sin(\log x)$, show that

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$$ (P.U. 1950)

(*iii*) If $y=\sin^{-1}x$, prove that
$(1-x^2)y_2-xy_1=0$ (P.U.)

(*iv*) If $y=(\sin^{-1}x)^2$, show that
$(1-x^2)y_2-xy_1=2$ (P.U. 1949)

6 (i) If $y=\log(1+\cos x)$, prove that $y_3+y_1y_2=0$

(*ii*) If $y=x^4\log x$, prove that $y_6+\dfrac{24}{x^2}=0$

(*iii*) $y^2=4a(x-b)$, show that $y\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^2=0.$

(P.U.)

9.9. Leibnitz's Theorem*. *The nth differential coefficient of the product of two functions.*

Let $y=uv$ where u and v are functions of x and let suffixes denote differentiation with regard to x. By differentiating successively. we have

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*Not included in the Punjab University Syllabus.

$$y_1=u_1v+uv_1$$
$$y_2=u_2v+2u_1v_1+uv_2$$
$$y_3=u_3v+3u_2v_1+3u_1v_2+uv_3$$
$$y_4=u_4v+4u_3v_1+6u_2v_2+4u_1v_3+uv_4$$

and so on.

It is obvious that the suffixes of u and v and the numerical co-efficients of the various terms correspond respectively to the indices of x and a and the co-efficients of the various terms in the expansion of $(x+a)^n$ by Binomial Theorem.

Assuming that this holds true for all the derivatives, we have

$$y_n=u_nv+nu_{n-1}v_1+\frac{n(n-1)}{2\,!}u_{n-2}v_2+\ldots\ldots$$
$$+\ldots\ldots+nu_1v_{n-1}+uv_n.$$

This is Leibnitz's Theorem for the nth derivative of the product of two functions and can easily be proved by the method of Induction.

Examples

Ex. 1. If $y=x^3e^{bx}$ find, y_n.

Let $u=e^{bx}$ and $v=x^3$

then $u_1=be^{bx},\ u_2=b^2e^{bx}\ldots\ldots,\ u_n=b_ne^{bx}$

and $v_1=3x^2,\ v_2=6x,\ v_3=6,\ v_4=v_5\ldots\ldots=0.$

By Leibnitz's Theorem, we have

$$y_n=u_nv+nu_{n-1}v_1+\frac{n(n-1)}{2\,!}u_{n-2}v_2+\ldots\ldots$$
$$=b^ne^{bx}.x^3+nb^{n-1}e^{bx}\,3x^2+\frac{n(n-1)}{2\,!}\,b^{n-2}e^{bx}.6x$$
$$+\frac{n(n-1)(n-2)}{3\,!}b^{n-3}\,e^{bx}.6+0$$
$$=b^{n-3}\,e^{bx}\,[b^3x^3+3nb^2x^2+3n(n-1)bx+n(n-1)(n-2)]$$

Ex. 2. If $y=x^2\sin x$, find y_n.

Let $u=\sin x$ and $v=x^2$, then $u_n=\sin\,(x+\frac{1}{2}n\pi)$.

and $v_1=2x,\ v_2=2,\ v_3=v_4=\ldots\ \ldots\ \ldots=0.$

$$\therefore\quad y_n=u_nv+nu_{n-1}v_1+\frac{n(n-1)}{2\,!}\,u_{n-2}\,v_2+\ldots\ \ldots\ \ldots.$$
$$=x^2\sin\,[x+\tfrac{1}{2}n\,\pi]+2nx\sin\,[x+\tfrac{1}{2}\,(n-1)\pi]$$
$$+n(n-1)\sin\,[x+\tfrac{1}{2}\,(n-2)\pi].$$

Ex. 3. Apply Leibnitz's theorem to find y_n for the following values of y :—

(*i*) xe^x. (*ii*) $x^2 e^{bx}$, (*iii*) x^4e^{3x}.
(*iv*) $x^3 \sin x$. (*v*) $x^3 \log x$.

Ex. 4. (*a*) Find $\dfrac{d^4y}{dx^4}$ when $y=\dfrac{e^{2x}}{x}$.

(*b*) Find by Leibnitz's Theorem $\dfrac{d^5y}{dx^5}$ when

(*i*) $y=x^2 \cos 3x$. (*ii*) $y=e^x \sin 2x$.

Verify Leibnitz's theorem by finding otherwise $\dfrac{d^5y}{dx^5}$.

Revision Questions IV

1. Find the limit of the following expressions :—

(*i*) $\dfrac{x^n-1}{x-1}$ when $x \to 1$.(*ii*) $\left(1+\dfrac{1}{x}\right)^x$ when $x \to \infty$.

(P. U. 1950)

(*ii*) $\left(1+\dfrac{a}{x}\right)^x$ when $x \to \infty$. (P. U.)

2. Show that the $\underset{h \to 0}{\text{Limit}} \dfrac{e^h-1}{h}=1$.

3. Differentiate the following with respect to x :—
(*i*) $y=\log \sin x$. (*ii*) $y^2=4ax$.
Verify in each case that $\dfrac{dy}{dx} \times \dfrac{dx}{dy}=1$. (P. U.)

4. Differentiate.
(*i*) $y=\sin [\log (\tan x)]$. (*ii*) $y=x \log \sin x$. (P.U.)

5. Show how to investigate the differential coefficients of functions of the form

$$f(x) \times \phi(x) \text{ and } [f(x)]^{\phi(x)}.$$ (B. U.)

[*Hint.* Put $f(x)=u$ and $\phi(x)=v$.].

6. Differentiate with respect to x :—

(*i*) $\sec x \tan x+\log \tan \left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$. (P. U.)

(*ii*) $\log \dfrac{e^x-e^{-x}}{e^x+e^{-x}}$. (*iii*) $\log (\cos x+\sqrt{-1} \sin x)$.

(*iv*) $xy(x-y)=2a^3$. (P. U.)

CHAPTER X.

EXPANSIONS

10·1. A **series** consists of a number of terms connected by the signs + and − written in a definite order, where each term is formed from the preceding one by some definite rule.

Familiar examples are *arithmetic series*, *geometric series* and series obtained by expansion of a binomial of the form $(x+a)^n$.

10·11. Finite and Infinite series. When the number of terms in a series is finite as in the case of $(x+a)^n$ when n is a positive integer, the series is said to be *finite*. It is said to be an *infinite* series when it is of the form

$$u_1+u_2+u_3+\ldots\ldots+u_n+\ldots\ldots\ldots$$

i. e. it consists of an unlimited number of terms and has no last term as is the case with $(x+a)^n$ when n is not a positive integer.

10·12. Sum of an Infinite series. The sum of a finite series is merely the algebraic sum of all the terms and can always be found by direct addition.

On the other hand, an infinite series has no sum in the ordinary sense of the term, since no matter how many terms we might add up there would always be an infinite number left over.

In this case by the sum of the series we mean the limit of sum of n terms of the series when n tends to infinity, so that,

Definition. *The sum of an infinite series* (s) *is the limit, if it exists, of the sum of the first* n *terms* (s_n) *of the series as* n *increases indefinitely, i.e.* $s=\underset{n\to\infty}{\mathrm{Lt}}\, s_n$.

When this limit exists and is finite, the series is said to *converge*, if it does not exist, the series is said to *diverge*.

In the elementary applications divergent series are of no importance. An infinite series has a definite value only when it converges. Before being able to use a given series, we must determine whether it converges or diverges.

10·13. Ratio test for convergence. If T_{n+1} be a general term of a series and T_n. the term preceding it, then if $\underset{n\to\infty}{\text{Lt}} \dfrac{T_{n+1}}{T_n} < 1$ numerically, the series is convergent.

This is only one of the tests for convergence and is applicable to most of the ordinary cases, *e. g.* for the well known geometric series

$$a+ar+ar^2+ar^3+ \dots\dots\dots$$

$T_{n+1}/T_n = ar^n/ar^{n-1} = r$, so that this is convergent when $-1<r<1$.

10·2. Let us consider the expansion of $\dfrac{1}{1-x}$ or $(1-x)^{-1}$. By the ordinary process of division in algebra, we obtain

$$\frac{1}{1-x} \equiv 1+x+\frac{x^2}{1-x} \qquad \dots\dots\dots(1).$$

$$\frac{1}{1-x} \equiv 1+x+x^2+\frac{x^3}{1-x} \qquad \dots\dots(2)$$

$$\frac{1}{1-x} \equiv 1+x+x^2+x^3+\frac{x^4}{1-x} \qquad \dots\dots\dots(3)$$

If the division is carried on further, we obtain other indentities and, in general,

$$\frac{1}{1-x} \equiv 1+x+x^2+ \dots \dots +x^{n-1}+\frac{x^n}{1-x} \qquad \dots\dots\dots(4)$$

The last term $x^n/(1-x)$ is called the *remainder* the *n*th term.

These equations give the expansions of $(1-x)^{-1}$ in finite number of terms and they are identities satisfied for all real values of x. For example, if we put $x=2$ in (3), we get

$$-1=1+2+4+8+\frac{16}{1-2}=15-16=-1.$$

10·21 We now consider the possibility of expanding $(1-x)^{-1}$ in series of positive integral powers of x by the method of calculus.

Assume. $(1-x)^{-1}=a+bx+cx^2+dx^3+ex^4+ \quad \ldots\ldots(1)$

Differentiating repeatedly, we have

$$(1-x)^{-2}=b+2cx+3dx^2+4ex^3+ \quad \ldots\ldots(2)$$

$$2(1-x)^{-3}=2c+6dx+12ex^2+ \quad \ldots\ldots(3)$$

Putting $x=0$ in each of these equations, we get

$a=1$, $b=1$, $2c=2$ *i.e.* $c=1$ and so on and thus we get

$$(1-x)^{-1}=1+x+x^2+x^3+x^4+ \ldots \ldots +x+x^{n+1}+ \ldots \quad (5)$$

This process appears to have no end and so we get an infinite series which has a value only when it is convergent.

In this case $\underset{n\to\infty}{\text{Lt.}} \frac{T_{n+1}}{T_n}=\underset{n\to\infty}{\text{Lt}} \frac{x^{n+1}}{x^n}=x.$

This is less than 1 numerically only if x is so *i.e.* if $-1<x<1$.

So that (5) holds true only when $x<1$ numerically. This can be seen by putting $x=2$ in (5), we get $-1=1+2+2^2+2^3+2^4+ \ldots = 1+2+4+8+16+ \ldots \ldots$ which is obviously absurd.

Thus if we assume $(1-x)^{-1}$ to be equal to an infinite series, it will hold only if $x<1$ numerically.

Hence, in general, if we assume a function $f(x)$ to be capable of being expanded in an infinite series of positive integral powers of x, the series will hold true only for those values of x, for which the series is convergent.

Note. Equation (5) could be obtained also by the application of Binomial theorem.

10·22. Particular cases of certain functions are met with in Algebra and Trigonometry. In calculus we obtain expansions with the help of an important theorem known as **Maclaurin's Theorem.**

It is also known as **Stirling's Theorem.** Calculus methods will be found simpler and more general.

10·3 Maclaurin's Theorem.

If $f(x)$ is capable of being expanded in an infinite series of positive integral powers of x then.

$$\mathbf{f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots\ldots\ldots\ldots}$$

where $f(0)$, $f'(0)$ $f''(0)$...are the values of $f(x)$ and its successive differential co-efficients for $x=0$.

If the function $f(x)$ is capable of being expanded in an infinite series of positive integral powers of x, let

$$f(x)=A+Bx+Cx^2+Dx^3+Ex^4+\ldots\ldots\ldots \quad (1)$$

where A,B,C......are constants whose values we have to determine.

Differentiating successively with respect to x we have

$$f'(x)=B+2Cx+3Dx^2+4Ex^3+\ldots\ldots\ldots \quad (2)$$
$$f''(x)=2C+2\cdot3Dx+3.4Ex^2+\ldots\ldots\ldots \quad (3)$$
$$f'''(x)=2.3D+2.3.4Ex+\ldots\ldots\ldots \quad (4)$$

and so on.

Since equations (1,) (2), (3) etc. are true for all values of x, they are also true for $x=0$.

Putting $x=0$ in the equations, we obtain

from (1) $f(0)=A$ *i.e.*, $A=f(0)$
from (2) $f'(0)=B$ or $B=f'(0)$

from (3) $f''(0)=2C$ or $C=\dfrac{f''(0)}{2!}$

from (4) $f'''(0)=2.3D$ or $D=\dfrac{f'''(0)}{3!}$ and so on.

Substituting these values of A,B,C...in (1), we have

$$f(x)=f(0)+x\,f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots\ldots\ldots\ldots$$

Note. This should not be regarded as a proof of Maclaurin's Theorem. A formal proof is beyond the scope of this book.

10·31 Failure of Maclaurin's Theorem. (*a*) This theorem fails to expand any function of x say, $f(x)$, when $f(x)$ or any of its successive derivatives does not exist or is infinite for $x=0$.

(*b*) Expansion of $f(x)$ by Maclaurin's Theorem also fails for those values of x for which the corresponding infinite series is not convergent.

For example, if $f(x)=\log x$, $f(0)$ does not exist and hence $\log x$ cannot be expanded in a series consisting of powers of x.

Again as seen before, $(1-x)^{-1}$ cannot be expanded if x is numerically greater than 1, for in that case the series is divergent.

Application of Maclaurin's Theorem

10.41. Expansion of sin. x.

Let

$$\begin{aligned} f(x)&=\sin x, & \therefore\quad f(0)&=0\\ f'(x)&=\cos x, & f'(0)&=1\\ f''(x)&=-\sin x, & f''(0)&=0\\ f'''(x)&=-\cos x, & f'''(0)&=-1\\ f^{\text{iv}}(x)&=\sin x, & f^{\text{iv}}(0)&=0\\ f^{\text{v}}(x)&=+\cos x, & f^{\text{v}}(0)&=1. \end{aligned}$$

By Maclaurin's Theorem, we have

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2\,!}f''(0)+\frac{x^3}{3\,!}f'''(0)+\ldots$$

Hence $$\sin x=x-\frac{x^3}{3\,!}+\frac{x^5}{5\,!}-\ldots\ldots\ldots \tag{1}$$

Differentiating both sides of this equation with respect to x, we have

$$\cos x=1-\frac{x^2}{2\,!}+\frac{x^4}{4\,!}-\frac{x^6}{6\,!}+\ldots\ldots\ldots \tag{2}$$

This can also be obtained directly from Maclaurin's Theorem.

Note. These series are convergent for all real values of x.

The terms are alternately positive and negative consisting of odd powers of x in the case of $\sin x$, and even powers of x in the case of $\cos x$.

10·42. Logarithmic Series. *Expansion of log* $(1+x)$.

Let $f(x)=\log(1+x)$. $f(0)=\log 1=0$

then $f'(x)=\dfrac{1}{1+x}=(1+x)^{-1}$, $f'(0)=1$

$$f''(x)=-(1+x)^{-2}, \quad f''(0)=-1$$
$$f'''(x)=2(1+x)^{-3}, \quad f'''(0)=2$$
$$f^{\text{iv}}(x)=-2.3(1+x)^{-4}, \quad f^{\text{iv}}(0)=-2.3$$
$$f^{\text{v}}(x)=2.3.4(1+x)^{-5}, \quad f^{\text{v}}(0)=2.3.4 \text{ and so on.}$$

By Maclaurin's Theorem, we have

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2\,!}f''(0)+\ldots\ldots$$

$$\therefore \log(1+x)=0+x+\frac{x^2}{2\,!}(-1)+\frac{x^3}{3\,!}(2)+\frac{x^4}{4\,!}(2.3)+\ldots$$

Hence $$\log(1+\mathbf{x})=\mathbf{x}-\frac{\mathbf{x}^2}{2}+\frac{\mathbf{x}^3}{3}-\frac{\mathbf{x}^4}{4}+\ldots\ldots$$

Note. The logarithmic series is convergent when $-1<x\leqslant 1$.

Cor. (*i*) $$\log(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\ldots\ldots$$

(*ii*) $$\log(1+x)^m=m\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\ldots\right)$$

10.43. Exponential Series. *Expansions of* a^x, e^x.

Let $f(x)=a^x \quad \therefore \quad f(0)=1$

then $f'(x)=a^x\log a \quad \therefore \quad f'(0)=\log a$

$$f''(x)=a^x(\log a)^2 \quad \therefore \quad f''(0)=(\log a)^2$$
$$f'''(x)=a^x(\log a)^3 \quad \therefore \quad f'''(0)=(\log a)^3$$

and so on.

By Maclaurin's theorem, we have

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2\,!}f''(0)+\frac{x^3}{3\,!}f'''(0)+\ldots\ldots$$

Hence $$a^x=1+x\log a+\frac{x^2}{2\,!}(\log a)^2+\frac{x^3}{3\,!}(\log a)^3+\ldots(1)$$

To obtain the expansion of e^x, we put $a=e$, so that
$$\log a=\log e=1.$$

Hence $$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3\,!}+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Cor. Putting $x=1$ in (2), we have

$$e=1+1+\frac{1}{2\,!}+\frac{1}{3\,!}+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Note 1. Equations (1) and (2) are true for every real value of x. For, $\frac{T_{n+1}}{T_n}=\frac{x^{n+1}}{(n+1)\,!}\div\frac{x^n}{n\,!}=\frac{x}{n+1}\to 0$ as $n\to\infty$, for all values of x, and thus the series (2) is convergent. Similar is the case (1).

2. Equation (2) can be obtained directly from Maclaurin's theorem.

10.44. Expansion of $\sin^{-1}x$.

Let $f(x)=\sin^{-1}x,$ $\qquad f(0)=0$

then $f'(x)=\frac{1}{\sqrt{1-x^2}}=(1-x^2)^{-\frac{1}{2}}$ $\qquad f'(0)=1$

$f''(x)=x(1-x^2)^{-\frac{3}{2}},$ $\qquad f''(0)=0$

$f'''(x)=(1-x^2)^{-\frac{3}{2}}+3x^2(1-x^2)^{-\frac{5}{2}}$ $\qquad f'''(0)=1$

$f^{iv}(x)=9x(1-x^2)^{-\frac{5}{2}}+15x^3(1-x^2)^{-\frac{7}{2}}$ $\qquad f^{iv}(0)=0$

$f^{v}(x)=9(1-x^2)^{-\frac{5}{2}}+\text{etc.}$ $\qquad f^{v}(0)=9$

and so on

Substituting in Maclaurin's Theorem, we have

$$\sin^{-1}x=0+x(1)+\frac{x^2}{2\,!}(0)+\frac{x^3}{3\,!}(1)+\frac{x^4}{4\,!}(0)+\frac{x^5}{5\,!}(9)+\ldots$$

Hence $\sin^{-1}x = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{x^5}{5} + \dots\dots$

Cor. Putting $x=\frac{1}{2}$ and multiplying by 2, we have

$$\frac{\pi}{3} = 1 + \frac{1}{8} \cdot \frac{1}{3} + \frac{1}{8} \cdot \frac{3}{16} \cdot \frac{1}{5} + \frac{1}{8} \cdot \frac{3}{16} \cdot \frac{5}{24} \cdot \frac{1}{7} + \dots$$

from which we get $\pi = 3{\cdot}14\dots\dots$

10·45. Expansion *of* $(a+x)^n$. *Binomial Theorem*

Here $f(x)=(a+x)^n$ $\qquad f(0)=a^n$

$f'(x)=n(a+x)^{n-1}$. $\qquad f'(0)=na^{n-1}$

$f''(x)=n(n-1)(a+x)^{n-2}$. $\qquad f''(0)=n(n-1)a^{n-2}$

and so on.

(*i*) When n is a positive integer, the nth diff. coeff. will be $n\,!$, a constant, and the differential coefficients of the higher order will vanish and thus the expansion will stop with the term x^n.

Hence by Maclaurin's Theorem

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2\,!}a^{n-2}x^2 + \dots + x^n.$$

(*ii*) When n is not a positive integer, the process of differentiation never comes to an end and hence we obtain the infinite series :—

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2\,!}a^{n-2}x^2 + \dots\dots$$

Note. Putting $a=1$, we get

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2\,!}x^2 + \dots\dots$$

This series is convergent when $-1 < x < 1$.

10·46. The series obtained so far are collected below for reference :—

1. $e^x = 1 + x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \dots\dots$
2. $a^x = 1 + x \log a + \frac{x^2(\log a)^2}{2\,!} + \frac{x^3(\log a)^3}{3\,!} + \dots\dots$
3. $\sin x = x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \frac{x^7}{7\,!} + \dots\dots$

4. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots$

5. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\ldots$

6. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$

7. $\sin^{-1}x = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1.3}{2.4} \cdot \frac{x^5}{5} + \frac{1.3.5}{2.4.6} \cdot \frac{x^7}{7} + \ldots\ldots$

8. $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots$

9. $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots\ldots$

Exercise XXIX

Apply Maclaurin's Theorem to the following expansions :—

1. (*i*) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$

(*ii*) $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots\ldots$

(*iii*) $e^{ax} = 1 + ax + \frac{a^2x^2}{2!} + \ldots\ldots$

(*iv*) $\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots$

(*v*) $2^x = 1 + x\log 2 + \frac{(x\log 2)^2}{2!} + \frac{(x\log 2)^3}{3!} + \ldots\ldots$

(*vi*) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\ldots$

(*vii*) $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots$

Deduce from this that $\pi = 3.14\ldots\ldots$

Obtain new expansions by differentiating each of these equations.

2. (*i*) $\sqrt{1+x}=1+\frac{1}{2}x-\frac{x^2}{2^2.2\,!}+\frac{3x^3}{2^3.3\,!}-\frac{3.5x^4}{2^4.4\,!}+\ldots\ldots$

(*ii*) $\frac{1}{\sqrt{1+x}}=1-\frac{1}{2}x+\frac{1.3}{2.4}x^2-\frac{1.3.5}{2.4.6}x^3+\ldots\ldots$

(*iii*) $\frac{1}{\sqrt{1-x^2}}=1+\frac{1}{2}x^2+\frac{1.3}{2.4}x^4+\frac{1.3.5}{2.4.6}x^6+\ldots\ldots$

x being numerically less than 1.

3. (*a*) Show that :

(*i*) $\log_a(1+x)=\left[x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\right]\log_a e$

(*ii*) $\log\sqrt{1-x^2}=-\frac{x^2}{2}-\frac{x^4}{4}-\frac{x^6}{6}-\frac{x^8}{8}-\ \ldots\ \ldots\ \ldots$

(*iii*) $\log\sqrt{\frac{1+x}{1-x}}=x+\frac{x^3}{3}+\frac{x^5}{5}+\ \ldots\ \ldots\ \ldots\ldots$

(*b*) Obtain the first few terms in the expansion of $\log_e\frac{1+x}{1-x}$ and hence calculate the value of $\log_e\frac{101}{99}$ correct to six places of decimals. (P.U.)

4. Expand $\tan x$ by Maclaurin's formula and verify by dividing the series for $\sin x$ by that for $\cos x$.

5. By expanding $\cos 2x$ by Maclaurin's series, show that

$$\sin^2x=\frac{1-\cos 2x}{2}=2\ \frac{x^2}{2\,!}-\frac{2^3x^4}{4\,!}+\frac{2^5x^6}{6\,!}+\ldots\ldots$$

Find also the expansion of $\cos^2x$ and hence verify that $\sin^2x+\cos^2x=1$.

6. Show that $(x-2)e^x+x+2$ is positive for all positive values of x.

7. Use Maclaurin's formula to prove that

$$\tan^{-1}(1+x)=\frac{\pi}{4}+\frac{x}{2}-\frac{x^2}{4}+\frac{x^3}{12}+\ldots \quad \text{(P.U.1952)}$$

8. If θ is the circular measure of an angle, so small that its 4th and higher powers may be neglected, show that

$$\sin\theta = \theta - \tfrac{1}{6}\theta^3 \text{ and } \cos\theta = 1 - \tfrac{1}{2}\theta^2$$

If a straight tunnel were to be bored through the earth from Delhi to Lucknow, a distance of 300 miles, how much distance would be saved ? [Radius of the earth = 4000 miles.]

9. Compute to four places of decimals :—
(*i*) sin 15°. (*ii*) sin 3°. (*iii*) cos 20°. (*iv*) tan 10°. (*v*) $\sqrt{e}$.

10. If $i = \sqrt{-1}$, show that $e^{i\theta} = \cos\theta + i\sin\theta$,

and hence deduce $e^{i\pi} = -1$.

Note. This is a wonderful result that $e^{i\pi}$ is an exact number equal to -1 whereas e is not exact being equal to an infinite series (see cor. § 10·43), π is not exact; and i is an imaginary number.

10·5 Further examples. We now add a few more examples on expansion by Maclaurins theorem.

Ex. 1 Show that

$$e^x \cos x = 1 + x - \frac{2x^3}{3!} - \frac{2^2x^4}{4\,!} - \frac{2^2x^5}{5!} + \frac{2^3x^7}{7!} + \ldots\ldots$$

Let $f(x) = e^x \cos x$, then $f(0) = 1$.
Now $f^n(x) = (\sqrt{2})^n e^x \cos(x + \frac{1}{4}n\pi)$
[Art. 9·71 when $a = b = 1$].

Putting $x = 0$ and giving to n values 1,2,3 etc, we obtain

$$f'(0) = \sqrt{2}\cos\tfrac{1}{4}\pi = 1, \qquad f''(0) = (\sqrt{2})^2 \cos\tfrac{1}{2}\pi = 0$$
$$f'''(0) = (\sqrt{2})^3 \cos\tfrac{3}{4}\pi = -2 \qquad f^{\text{iv}}(0) = (\sqrt{2})^4 \cos\pi = -2^2$$
$$f^{\text{v}}(0) = (\sqrt{2})^5 \cos\tfrac{5}{4}\pi = -2^2, \qquad f^{\text{vi}}(0) = (\sqrt{2})^6 \cos\tfrac{3}{2}\pi = 0$$
$$f^{\text{vii}}(0) = (\sqrt{2})^7 \cos\tfrac{7}{4}\pi = 2^3\sqrt{2}\cos(-\tfrac{1}{4}\pi) = 2^3.$$

$\therefore$ by Maclaurin's theorem, we have

$$e^x \cos x = f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots\ldots\ldots\ldots\ldots$$
$$= 1 + x - \frac{2x^3}{3!} - \frac{2^2x^4}{4!} - \frac{2^2x^5}{5!} + \frac{2^3x^7}{7!} + \ldots\ldots\ldots$$

Ex. 2. Apply Maclaurin's theorem to expand $e^{\sin x}$ in powers of x upto 4 terms. (P.U. 1950)

Let $f(x)=e^{\sin x}$ then $f(0)=1$

$f'(x)=e^{\sin x}\cos x$ $\qquad f'(0)=1$

$f''(x)=e^{\sin x}(\cos^2 x-\sin x)$ $\qquad f''(0)=1$

$$f'''(x)=e^{\sin x}\cos x\,(\cos^2 x-\sin x)+e^{\sin x}(-2\cos x\sin x-\cos x)$$

$$=e^{\sin x}(\cos^3 x-3\sin x\cos x-\cos x),\quad f'''(0)=0$$

$f^{\text{iv}}(x)=e^{\sin x}(\cos^4 x-4\cos^2 x+$terms involving $\sin x)$

$\therefore\ f^{\text{iv}}(0)=-3.$

Hence by Maclaurin's theorem, we have

$$e^{\sin x}=f(x)=f(0)+xf(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots\ldots$$

$$=1+x+\frac{x^2}{2!}(1)+\frac{x^3}{3!}(0)+\frac{x^4}{4!}(-3)+\ \ldots\ldots\ldots$$

$$=1+x+\tfrac{1}{2}x^2-\tfrac{1}{8}x^4+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

10.51 It is sometimes more convenient to obtain the expansion of more complicated functions with the help of known standard series, as given in Art. 10·46.

Ex. 1. Expand $e^{\sin x}$ up to four terms

Let $z=\sin x=x-\frac{1}{6}x^3+\ldots\ldots$

then $e^{\sin x}=e^z=1+z+\frac{1}{2}z^2+\frac{1}{6}z^3+\frac{1}{24}z^4+\ldots\ldots$

$$=1+(x-\tfrac{1}{6}x^3+\ldots)+\tfrac{1}{2}(x-\tfrac{1}{6}x^3\ldots)^2$$
$$+\tfrac{1}{6}(x-\ldots)^3+\tfrac{1}{24}(x-\ldots)^4+\ldots$$
$$=1+x-\tfrac{1}{6}x^3+\tfrac{1}{2}(x^2-\tfrac{1}{3}x^4)+\tfrac{1}{6}x^3+\tfrac{1}{24}x^4\ldots$$
$$=1+x+\tfrac{1}{2}x^2+(\tfrac{1}{6}-\tfrac{1}{6})x^3-(\tfrac{1}{6}-\tfrac{1}{24})x^4+\ldots$$
$$=1+x+\tfrac{1}{2}x^2-\tfrac{1}{8}x^4\ldots$$

Note. Campare Ex. 2.. Art. 10.5 where this expansion is obtained directly by Maclaurin's theorem.

Ex. 2. Prove that

$$\log(1-x+x^2)$$
$$=-x+\tfrac{1}{2}x^2+\tfrac{2}{3}x^3+\tfrac{1}{4}x^4-\tfrac{1}{5}x^5-\tfrac{1}{3}x^6-\tfrac{1}{7}x^7+\tfrac{1}{8}x^8\ldots \quad \text{(P.U.)}$$

We have, $\log(1-x+x^2)$

$$=\log(1+x^3)/(1+x)=\log(1+x^3)-\log(1+x)$$
$$=(x^3-\tfrac{1}{2}x^6+\ldots)-(x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\tfrac{1}{5}x^5-\tfrac{1}{6}x^6$$
$$+\tfrac{1}{7}x^7-\tfrac{1}{8}x^8+\ldots)$$

$$=-x+\tfrac{1}{2}x^2+(1-\tfrac{1}{3})x^3+\tfrac{1}{4}x^4-\tfrac{1}{5}x^5-(\tfrac{1}{2}-\tfrac{1}{6})x^6-\tfrac{1}{7}x^7+\tfrac{1}{8}x^8+\dots$$
$$=-x+\tfrac{1}{2}x^2+\tfrac{2}{3}x^3+\tfrac{1}{4}x^4-\tfrac{1}{5}x^5-\tfrac{1}{3}x^6-\tfrac{1}{7}x^7+\tfrac{1}{8}x^8+\dots$$

Ex. 3. Expand $(1+\sin x)^x$ as far as the term containing x^4.

Let $y=(1+\sin x)^x$, then

$$\log y=x\log(1+\sin x)=x[\sin x-\tfrac{1}{2}\sin^2 x+\tfrac{1}{3}\sin^3 x+\dots]$$
$$=x(x-\tfrac{1}{6}x^3+\dots)-\tfrac{1}{2}x(x-\tfrac{1}{6}x^3+\dots)^2+\tfrac{1}{3}x(x-\tfrac{1}{6}x^3+\dots)^3+\dots$$
$$=x^2-\tfrac{1}{6}x^4-\tfrac{1}{2}(x^3+\dots)+\tfrac{1}{3}(x^4+\dots)$$
$$=x^2-\tfrac{1}{2}x^3+\tfrac{1}{6}x^4=z, \text{ say,}$$

$$\therefore\quad y=e^z=1+z+\tfrac{1}{2}z^2+\dots\dots$$
$$=1+(x^2-\tfrac{1}{2}x^3+\tfrac{1}{6}x^4+\dots)+\tfrac{1}{2}(x^2-\tfrac{1}{2}x^3+\tfrac{1}{6}x^4+\dots)^2$$
$$=1+x^2-\tfrac{1}{2}x^3+\tfrac{1}{6}x^4+\tfrac{1}{2}x^4+\dots$$

Hence $(1+\sin x)^x=1+x^2-\tfrac{1}{2}x^3+\tfrac{2}{3}x^4+\dots\dots$

Exercise XXX

Apply Maclaurin's theorem to expand.

1 (*i*) $\sec x$ (P.U. 1949)
(*ii*) $\log\cos x$ (P.U. 1948)

and deduce that

(*iii*) $\log\sec x=\dfrac{x^2}{2\,!}+\dfrac{2x^4}{4\,!}+\dfrac{16x^6}{6\,!}+\dots\dots$

(*iv*) $e^x\sec x=1+x+x^2+\tfrac{2}{3}x^3+\tfrac{1}{2}x^4+\dots$

2 (*i*) $\log(1+\sin x)$ (P.U. 1948)

[**Hint.** First diff. co-eff.$=\sec x-\tan x$]

(*ii*) $\log\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$ (P.U.)

[**Hint.** Ist. diff co-eff.$=\sec x$]

(*iii*) $\log(1+e^x)$ (P.U.) (*iv*) $\log(1+\tan x)$ (P.U.)

3 Expand the following :—

(*i*) $\log(1+x+x^2)$ (P.U.) (*ii*) $\log(1+4x+3x^2)$

(*iii*) $(1+x)^{1+x}$ (P.U. 1943) (*iv*) $\log\dfrac{\tan x}{x}$ (P.U. 1952)

(*v*) $e^x\sin x$ (*vi*) $e^x/(1+e^x)$ (P.U.)

4. By using the expansions of the functions involved, find the limit of the following when $x \to 0$:

(*i*) $\dfrac{xe^x - \log(1+x)}{x^5}$ (*ii*) $\dfrac{x\cos x - \log(1+x)}{x^2}$

(*iii*) $\dfrac{\sin x - x + \frac{1}{6}x^3}{x^5}$ (*iv*) $\dfrac{a^x - b^x}{x}$

5. Find the values of the constants a and b, so that the expansion of function

$$x + a\tan^{-1}x - b\sin x$$

in ascending power of x, may begin with the term in x^5.

6. Obtain the expansion of a^x from that of e^x.

$$\left[\textbf{Hint. } a^x = e^{\log a^x} = e^{x\log a}\right].$$

10.6. If $f(x)$ be a function of x and if x is increased by h, where h is small, the approximate value of $f(x+h)$, when the value of $f(x)$ is known, is obtained by expanding $f(x+h)$ in a series of positive integral powers of h. This is done by *Taylor's theorem* which we now discuss.

10.7. Taylor's Theorem. *If $f(x+h)$ is capable of being expanded in an infinite series of positive integral powers of h, then,*

$$\mathbf{f(x+h) = f(x) + hf'(x) + \frac{h^2}{2\,!}f''(x) + \frac{h^3}{3\,!}f'''(x) + \ldots\ldots}$$

where $f(x)$ is the function obtained by putting $h=0$ in $f(x+h)$ and $f'(x)$, $f''(x)$ etc. are the successive differential co-efficients of $f(x)$.

Let $f(x+h) = A + Bh + Ch^2 + Dh^3 + Eh^4 \ldots\ldots\ldots\ldots\ldots\ldots$(1)

Differentiating successively with respect to h, (regarding x as constant). we have

$$f'(x+h) = B + 2Ch + 3Dh^2 + 4Eh^3 + \ldots\ldots\ldots\ldots\ldots \quad (2)$$
$$f''(x+h) = 2C + 2.3Dh + 3.4Eh^2 + \ldots\ldots \quad (3)$$
$$f'''(x+h) = 2.3D + 2.3.4Eh + \ldots\ldots \quad (4)$$

and so on

Putting $h=0$ in these equations, we have

from (1) $f(x)=A$ or $A=f(x)$
from (2) $f'(x)=B$ or $B=f'(x)$
from (3) $f''(x)=2C$ or $C=f''(x)/2\,!$
from (4) $f'''(x)=2\cdot 3D$ or $D=f'''(x)/3!$

Substituting these values of A, B, C etc., in (1), we have

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2\,!}f''(x)+\frac{h^3}{3\,!}f'''(x)+\ldots \quad (A)$$

10.71. Putting $x=a$, we have

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2\,!}f''(a)+\ldots\ldots\ldots \quad (B)$$

10·72. Expansion of f(x) in powers of x−a.

$f(x)$ can be written as $f[a+(x-a)]$.

By putting $x-a$ for h in (B) [or by putting $x=a$ and $h=x-a$ in (A)], we have

$$f(x)=f(a\ \overline{x-a})=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2\,!}f''(a)+\ldots (C)$$

This gives the expansion of the function in the neighbourhood of $x=a$. To compute $f(x)$ by this formula, we must know the values of $f(a)$, $f'(a)$, $f''(a)$ etc.

10·73. Failure of Taylor's Theorem.

Expansion by Taylor's Theorem is not valid :—

(*i*) when the series is not convergent

(*ii*) when $f(x)$ or any of its successive differential co-efficients does not exist or is infinite for values of x lying between x and $(x+h)$.

10·74. Maclaurin's Theorem deducible from Taylor's Theorem.

In the theorem

$$f(x+h)=f(x)+hf''(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3\,!}f'''(x)+\ldots\ldots$$

Putting $x=0$, we get,

$$f(h)=f(0)+hf'(0)+\frac{h^2}{2\,!}f''(0)+\frac{h^3}{3\,!}f'''(0)+\ldots\ldots$$

which is Maclaurin's Theorem with the variable h.

By changing h to x, we have

$$f(x)=f(0)+x\,f'(0)+\frac{x^2}{2\,!}f''(0)+\frac{x^3}{3\,!}f'''(0)\ \ldots\ \ldots$$

Application of Taylor's Theorem

10·81. *Expansion of a^{x+h} in powers of h.*

Let $f(x+h)=a^{x+h}$

then $f(x)=a^x$, $f'(x)=a^x\log a$, $f''(x)=a^x(\log a)^2$ etc.

$\therefore$ by Taylor's theorem, we have

$$a^{x+h}=f(x+h)=f(x)+hf'(x)+\frac{h^2}{2\,!}f''(x)+\ldots\ldots$$

$$=a^x+ha^x\log a+\frac{h^2}{2\,!}a^x(\log a)^2+\ldots$$

$$=a^x[1+h\log a+\frac{h^2}{2\,!}(\log a)^2+\ldots] \quad \text{(D)}$$

10·82. Putting $a=e$ in (D), we obtain

$$e^{x+h}=e^x[1+h+\frac{h^2}{2!}+\ldots\ldots] \quad \text{(E)}$$

This can also be obtained directly from Taylor's theroem.

Ex. Show that a^x can be expanded by Taylor's theorem. (P. U.)

[**Hint**. Put $x=0$ and $h=x$ in (D)].

10·83. *Expansion of log sin $(x+h)$ in powers of h.*

Let $f(x+h)=\log\sin(x+h)$, then $f(x)=\log\sin x$

$\therefore f'(x)=\cot x$, $f''(x)=-\operatorname{cosec}^2 x$,

$f'''(x)=+2\operatorname{cosec}^2 x\cot x$

Hence by Taylor's theorem, we have

$$\log\sin(x+h)=f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\ldots.$$

$$=\log\sin x+h\cot x-\frac{h^2}{2}\operatorname{cosec}^2 x$$

$$+\frac{h^3}{3}\cot x\operatorname{cosec}^2 x-\ldots\ldots$$

10·84. *Expansion of sin $(x+y)$ in powers of y.*

Let $f(x+y)=\sin(x+y)$ then $f(x)=\sin x$.

$\therefore$ $f'(x)=\cos x$, $f''(x)=-\sin x$, $f'''(x)=-\cos x$
$f^{\text{iv}}(x)=\sin x$, $f^{\text{v}}(x)=\cos x$, etc.

$\therefore$ by Taylor's Theorem, we have

$$\sin(x+y)=f(x+y)=f(x)+y\,f'(x)+\frac{y^2}{2!}f''(x)+\ldots\ldots\ldots$$

$$=\sin x+y\cos x-\frac{y^2}{2!}\sin x-\frac{y^3}{3!}\cos x$$

$$+\frac{y^4}{4!}\sin x+\quad\ldots\qquad\text{(F)}$$

Ex. 1. Expand sin $(x+y)$ in powers of y and deduce that $\sin(x+y)=\sin x\cos y+\cos x\sin y$.

Separating sin x and cos x in (F), we have

$$\sin(x+y)=\sin x\left[1-\frac{y^2}{2!}+\frac{y^4}{4!}-\ldots\right]$$

$$+\cos x\left[y-\frac{y^3}{3!}+\frac{y^5}{5!}-\ldots\right]$$

$$=\sin x\cos y+\cos x\sin y.$$

Ex. 2. Expand log x in the neighbourhood of $x=1$.

Here we have to expand $\log x$ in powers of $(x-1)$.

We have $f(x)=\log x$ and $f(1)=\log 1=0$.

$f'(x)=1/x$, $f'(1)=1$; $f''(x)=-1/x^2$, $f''(1)=-1$
$f'''(x)=2/x^3$, $f'''(1)=2$, $f^{\text{iv}}(x)=-6/x^4$, $f^{\text{iv}}(1)=-6$

Now by taking $x=1$ and $h=x-1$ in Taylor's theorem, we have

$$f(x)=f(1+\overline{x-1})=f(1)+(x-1)f'(1)+\frac{(x-1)^2}{2!}f''(1)+\ldots$$

$$\therefore \log x=\log[1+(x-1)]=\log 1+(x-1).1$$
$$+\frac{(x-1)^2}{2!}(-1)+\frac{(x-1)^3}{3!}(2)+\ldots\ldots$$
$$=(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\ldots.$$

Note. This can be written down by (C) Art 10·72, by taking $a=1$.

Ex. 2. Expand e^x in powers of $x-a$.

By (E) Art. 10·82, we have

$$e^{x+h}=e^x\left[1+h+\frac{h^2}{2!}+\frac{h^3}{3!}+\ldots\ldots\right]$$

Putting $x=a$ and $h=x-a$, we get

$$e^x=e^{a+\overline{x-a}}=e^a\left[1+(x-a)+\frac{(x-a)^2}{2!}+\frac{(x-a)^3}{3!}+\ldots\right]$$

Ex. 2. Deduce the expansion of $\sin x$ and $\cos y$ from that of $\sin(x+y)$.

[Hint. Put $y=0$ and $x=\frac{1}{2}\pi$ in (F) Art. 10·84].

Exercise XXXI

Apply Taylor's formula to obtain the following expansions :—

1. (*i*) $e^{x+h}=e^x\left\{1+h+\frac{h^2}{2!}+\frac{h^3}{3!}+\ldots\ldots\ldots\right\}$

(*ii*) $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots\ldots$

(*iii*) $\cos(x+h)=\cos x-h\sin x-\frac{h^2}{2!}\cos x+\frac{h^3}{3}\sin x-\ldots\ldots\ldots$

(*iv*) $\log(x+h)=\log x+\frac{h}{x}-\frac{h^2}{2x^2}+\frac{h^3}{3x^3}-\ldots\ldots\ldots$

(*v*) $\tan(x+h)=\tan x+h\sec^2 x+h^2\sec^2 x\tan x+\frac{h^3}{3}\sec^2 x(1+3\tan^2 x)\ldots\ldots$ (P.U.)

2. (*i*) $\log \sin (x-h)=\log \sin x+h \cot x-\frac{h^2}{2} \operatorname{cosec}^2 x$

$+\frac{h^3}{3} \cot x \operatorname{cosec}^2 x- \ldots\ldots$ (P.U.)

(*ii*) $\tan^{-1}(x+h)=\tan^{-1}x$

$$+\frac{h}{1+x^2}-\frac{xh^2}{(1+x^2)^2}-\frac{1-3x^2}{(1+x^2)^3}\cdot\frac{h^3}{3}-\ldots$$

(*iii*) $(x+h)^n=x^n+nhx^{n-1}+\frac{n(n-1)}{2\,!}h^2x^{n-2}+\ldots\ldots\ldots$

Distinguish between the two cases when n is a positive integer and when n is not a positive integer.

3. Expand :—

(*i*) e^x in powers of $x-2$. (D.U.)
(*ii*) $3x^3-5x^2-6x+9$ in powers of $x-3$.
(*iii*) $\tan x$ in powers of $x-\frac{1}{4}\pi$.
(*iv*) $\cos x$ in powers of $x-2$.
(*v*) $\tan^{-1}x$ in the neighbourhood of $x=1$.

4. Deduce from Taylor's Theorem :

(*a*) $\frac{1}{2}[f(x+h)+f(x-h)]=f(x)+\frac{h^2}{2!}f''(x)+\frac{h^4}{4!}f^{iv}(x)+\ldots$

(*b*) $f(x)=f(0)+xf'(x)-\frac{x^2}{2!}f''(x)+\frac{x^3}{3!}f'''(x)+\ldots\ldots\ldots$

(Bernoulli's series)

5. Expand.

(*i*) $\cos (a+x)$ in powers of x, and
(*ii*) $\sin x$ in powers of $x-a$.

10·85. Numerical calculation by Taylor's Theorem.

Ex. 1. $f(x)=x^3+x^2+15x-24$. Calculate $f(\frac{11}{10})$ by expanding $f(x+h)$ in powers of h and then putting suitable values for x and h. (P. U.)

$f(x)=x^3+8x^2+15x-24$ $\qquad f(1)=0$
$f'(x)=3x^2+16x+15$ $\qquad f'(1)=34$
$f''(x)=6x+16$ $\qquad f''(1)=22$
$f'''(x)=6$ $\qquad f'''(1)=6$

To obtain $f(\frac{11}{10})$, *i.e.*, $f(1+\frac{1}{10})$ put $x=1$ and $h=\frac{1}{10}$ in Taylor's series.

$$f\left(1+\frac{1}{10}\right)=f(1)+\frac{1}{10}f'(1)+\frac{1}{2!.10^2}f''(1)+\frac{1}{3!.10^3}f'''(1)$$
$$-\ldots\ldots\ldots$$
$$=0+\frac{1}{10}(34)+\frac{1}{2.10^2}(22)+\frac{1}{6.10^3}(6)$$
$$=3{\cdot}4+{\cdot}11+{\cdot}001$$
$$=3{\cdot}511.$$

Ex. 2. Given $\sin 30^\circ=\frac{1}{2}$, apply Taylor's Theorem to calculate the value of $\sin 31^\circ$. (P.U. 1950)

We have $30^\circ=\frac{1}{6}\pi$ radian and

$$1^\circ=\frac{\pi}{180}={\cdot}0175 \text{ radian.}$$

Also $\sin\frac{1}{6}\pi=\frac{1}{2}=0{\cdot}5$ and
$\cos\frac{1}{6}\pi=\sqrt{3}/2=0{\cdot}8660.$

Now by Taylor's Theorem (See Art. 10·84, y being changed to h),
we have,

$$\sin(x+h)=\sin x+h\cos x-\frac{h^2}{2}\sin x-\ldots\ldots\ldots$$

$$\therefore \sin 31^\circ=\sin\left(\frac{\pi}{6}+\frac{\pi}{180}\right)$$
$$=\sin\frac{\pi}{6}+\frac{\pi}{180}\cos\frac{\pi}{6}-\frac{1}{2}\left(\frac{\pi}{180}\right)^2\sin\frac{\pi}{6}\ldots\ldots$$
$$={\cdot}5+{\cdot}0175\times{\cdot}866-\tfrac{1}{2}({\cdot}0175)^2\times{\cdot}5-\ldots\ldots\ldots$$
$$={\cdot}5+{\cdot}015155-{\cdot}000076-\ldots\ldots\ldots\ldots\ldots$$
$$={\cdot}5150 \text{ upto four places of decimals.}$$

Exercise XXXII.

Apply Taylor's Theorem to calculate the values of :—

1. (*i*) $f(\frac{11}{10})$ if $f(x)=x^3+3x^2+15x-24$. (P.U. 1951)
(*ii*) $f(2{\cdot}001)$ if $f(x)=x^3-2x+5$. (P.U. 1952)
(*iii*) $f(1{\cdot}05)$ if $f(x)=x^3-6x^2+2x+1$.

2. (*i*) sin 30° 12′.
(*ii*) sin 29° 48′.
(*iii*) cos 30° 12′.
(*iv*) sin 1° 48′.

3. Expand by Maclaurin's Theorem $\tan\left(\frac{\pi}{4}+x\right)$ as far as the term in x^4. Hence calculate the value of tan 46° to five places of decimals. (P.U.)

CHAPTER XI

INTEGRATION

11·1 Integration as inverse of differentiation. We have been occupied upto this point with the problem of finding the derivatives of functions. Many of the most important applications of the calculus lead to the inverse problem, namely, that of *finding the function* (y) when its derivative $\left(\dfrac{dy}{dx}\right)$ is given. The required function is called an **integral** of the given derivative, and the process of finding it is called **integration.** The given function whose integral is to be found, is known as the **integrand**.

Suppose for example, $\dfrac{dy}{dx}=2x$, then $y=x^2$

i. e. x^2 is the integral of $2x$.

Again since $\dfrac{d}{dx}\left(\frac{1}{4}x^4\right)=\frac{1}{4}\times 4x^3=x^3$.

$\therefore$ the integral of x^3 is $\frac{1}{4}x^4$.

In general since $\dfrac{d}{dx}\left(\dfrac{x^{n+1}}{n+1}\right)=x^n$,

$\therefore$ the integral of x^n is $\dfrac{x^{n+1}}{n+1}$ $(n\neq-1)$......... (1).

Hence x^n *may be integrated by adding 1 to the exponent and dividing by the increased exponent.*

Note. Here n is not equal to -1, for in that case $n+1=0$ which leads to the impossible process of division by zero.

Since $\dfrac{d}{dx}(\log x)=\dfrac{1}{x}$, $\therefore$ **integral of** $\dfrac{1}{x}$ **is log x.** .... (2)

Note. This is the exceptional case mentioned in (1)

$\because \frac{d}{dx}(\sin x)=\cos x$, $\therefore$ **integral of** $\cos x$ **is** $\sin x$ ...(3).

$\because \frac{d}{dx}(\frac{1}{3}\sin 3x)=\cos 3x$ $\therefore$ integral of $\cos 3x$ is $\frac{1}{3}\sin 3x$.

$\because \frac{d}{dx}(-\cos x)=\sin x$, $\therefore$ **integral of** $\sin x$ **is** $-\cos x$ (4)

$\because \frac{d}{dx}(\tan x)=\sec^2 x$, $\therefore$ **integral of** $\sec^2 x$ **is** $\tan x$...(5)

$\because \frac{d}{dx}(e^x) \quad =e^x$ $\therefore$ **integral of** e^x **is** e^x (6)

$\because \frac{d}{dx}(\frac{1}{5}\ e^{5x}) = e^{5x}$ $\therefore$ integral of e^{5x} is $\frac{1}{5}e^{5x}$.

$\because \frac{d}{dx}(\frac{1}{a}e^{ax})=e^{ax}$ $\therefore$ **integral of** e^{ax} **is** $\frac{1}{a}e^{ax}$ (7)

If $\frac{d}{dx}[f(x)] = \frac{1}{x}$, then $f(x)=\log x$.

Examples

Find y for the following of values of $\frac{dy}{dx}$ and verify the results by differentiation :

(*i*) 1 (*ii*) 0 (*iii*) x (*iv*) $1+x$ (*v*) $1+x+x^2$

(*vi*) $5x^4$ (*vii*) x^{11} (*viii*) $\frac{5}{x}$ (*ix*) $x^5-\frac{1}{5x}$ (*x*) $\sin\theta$

(*xi*) $\cos 2\theta$ (*xii*) $\sec^2\theta$ (*xiii*) $\operatorname{cosec}^2 x$ (*xiv*) e^{3x}

(*xv*) $\sin 3x+e^{3x}$ (*xvi*) $\sin x+\cos x$ (*xvii*) $58x^{57}$

(*xviii*) $x^{\frac{7}{3}}$ (*xix*) $x^{-\frac{7}{3}}$.

11·2 Notation. If $\frac{d}{dx}[\phi(x)]=f(x)$, then $\phi(x)$ is said to be the *indefinite integral* or simply the *integral* of $f(x)$ with respect to x and is written symbollically as

$$\int f(x)dx=\phi(x).$$

The symbol $\int$, is called the *sign of integration* and $\int f(x)\,dx$ asks the question ; *"what function of x has f(x) as its differential coefficient"*? and is read as the integral of $f(x)$ with respect to x.

11.21 Unlike Differential Calculus there are no infallible rules in this subject by which we can discover a function, say $\phi(x)$, whose differential coefficient with respect to x is $f(x)$. Integration is largely a tentative process in which we can be guided only by the results of the direct process of differentiation.

For example, to find the integral of $\cos x$, we have to discover the function whose derivative is $\cos x$. From our knowledge of Differential Calculus, we know that such a function is $\sin x$, thus

$$\text{Since } \frac{d}{dx}(\sin x)=\cos x. \qquad \therefore \int \cos x\,dx=\sin x$$

$$\text{Also since } \frac{d}{dx}[\sin(x+a)]=\cos(x+a)$$

$$\therefore \int \cos(x+a)dx=\sin(x+a).$$

$$\text{Similarly } \int 3x^2dx=x^3 \text{ because} \frac{d}{dx}(x^3)=3x^2.$$

$$\text{Also } \int 3(x+a)^2dx=(x+a)^3$$

$$\text{because } \frac{d}{dx}(x+a)^3=3(x+a)^2$$

and so on. The following results may be similarly verified by differentiation :

$$\int \frac{1}{x}dx=\log x, \int \frac{dx}{x+a}=\log(x+a)$$

$$\int e^x\,dx=e^x, \quad \int e^{x+a}\,dx=e^{x+a}.$$

In general if the integral of a function of x, say $f(x)$ is $\phi(x)$, then the integral of $f(x+a)$ is $\phi(x+a)$.

$$i.e., \text{ if } \int f(x)dx=\phi(x), \text{ then } \int f(x+a)dx=\phi(x+a) \ldots \text{I}$$

Examples

Find the following indefinite integrals and verify the results by differentiation :—

(*i*) $(x+9)^{11}$. (*ii*) $\frac{5}{x+5}$. (*iii*) $\sin\left(\theta+\frac{1}{4}\pi\right)$.

(*iv*) $\sec^2(\theta+\frac{1}{5}\pi)$. (*v*) e^{x+7}. (*vi*) $\operatorname{cosec}^2(x+5\pi)$

(*vii*) $5(x+8)^9$ (*viii*) $5(x+c)^{75}$.

11.3 The first step towards integration is to write down a list of differentiation of various simple functions each of which will furnish us a corresponding result in integration. The student is advised to make himself thoroughly familiar with these results. The success in integration depends upon the knowledge of the results of differentiation.

Table of Preliminary Results on integration

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right)=x^n. \quad \therefore \quad \int x^n\,dx=\frac{x^{n+1}}{n+1}. \quad \ldots(1)$$

(except for $n=-1$).

$$\frac{d}{dx}\log_e x=\frac{1}{x} \quad \therefore \quad \int \frac{1}{x}\,dx=\log_e x \quad \ldots(2)$$

$$\frac{d}{dx}\left(\frac{a^x}{\log a}\right)=\frac{a^x\log a}{\log a}=a^x \quad \therefore \quad \int a^x\,dx=\frac{a^x}{\log_e a} \quad \ldots(3)$$

$$\frac{d}{dx}e^x=e^x, \quad \therefore \quad \int e^x\,dx=e^x. \quad \ldots(4)$$

$$\frac{d}{dx}\sin x=\cos x. \quad \therefore \quad \int \cos x\,dx=\sin x. \quad \ldots(5)$$

$$\frac{d}{dx}\cos x=-\sin x \quad \therefore \quad \int \sin x\,dx=-\cos x. \quad \ldots(6)$$

$$\frac{d}{dx}\tan x=\sec^2 x, \quad \therefore \quad \int \sec^2 x\,dx=\tan x. \quad \ldots(7)$$

$$\frac{d}{dx}\cot x=-\operatorname{cosec}^2 x \quad \therefore \quad \int \operatorname{cosec}^2 x\,dx=-\cot x. \quad \ldots(8)$$

$$\frac{d}{dx}\sec x=\sec x\tan x,\ \therefore\ \int\sec x\tan x\,dx=\sec x \ldots\ (9)$$

$$\frac{d}{dx}\operatorname{cosec} x=-\operatorname{cosec} x\cot x,\int\operatorname{cosec} x\cot x\,dx=-\operatorname{cosec} x \ldots(10)$$

$$\frac{d}{dx}\log\sin x=\frac{\cos x}{\sin x}=\cot x,\quad\therefore\int\cot x\ dx=\log\sin x \ldots(11)$$

$$\frac{d}{dx}\log\sec x=\frac{\sec x\tan x}{\sec x}=\tan x,$$

$$\therefore\quad\int\tan x\,dx=\log\sec x=-\log\cos x\quad\ldots(12)$$

$$\frac{d}{dx}\log\ \operatorname{cosec}\ x=\frac{-\operatorname{cosec} x\cot x}{\operatorname{cosec} x}=-\cot x,$$

$$\therefore\int\cot x\,dx=-\log\operatorname{cosec} x=\log\sin x\quad\ldots(13)$$

$$\left.\begin{aligned}\frac{d}{dx}\sin^{-1}x&=\frac{1}{\sqrt{1-x^2}}\\ \frac{d}{dx}\cos^{-1}x&=-\frac{1}{\sqrt{1-x^2}}\end{aligned}\right\}\ \therefore\int\frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x \text{ or } =-\cos^{-1}x\ldots(14)$$

$$\left.\begin{aligned}\frac{d}{dx}\tan^{-1}x&=\frac{1}{1+x^2}\\ \frac{d}{dx}\cot^{-1}x&=-\frac{1}{1+x^2}\end{aligned}\right\}\ \therefore\int\frac{dx}{1+x^2}\cancel{dx}=\tan^{-1}x \text{ or } =-\cot^{-1}x.\ \ldots(15)$$

$$\left.\begin{aligned}\frac{d}{dx}\sec^{-1}x&=\frac{1}{x\sqrt{x^2-1}}\\ \frac{d}{dx}\operatorname{cosec}^{-1}x&=-\frac{1}{x\sqrt{x^2-1}}\end{aligned}\right\}\ \therefore\int\frac{dx}{x\sqrt{x^2-1}}\cancel{dx}=\sec^{-1}x \text{ or } =-\operatorname{cosec}^{-1}x\ \ldots(16)$$

11·31. $\int(ax+b)^n dx.\quad (n\neq-1)$

$(ax+b)^n$ being a function of $ax+b$, suggests a trial differentiation of $(ax+b)^{n+1}$.

We have, $\frac{d}{dx}(ax+b)^{n+1}=a(n+1)(ax+b)^n$, which can be written as $\frac{d}{dx}\left[\frac{(ax+b)^{n+1}}{a(n+1)}\right]=(ax+b)^{n}$.

Hence we get, $\int \mathbf{(ax+b)^{n}dx=\frac{(ax+b)^{n+1}}{a(n+1)}}$. ✓

Thus $(ax+b)^{n}$ *may be integrated by adding 1 to the exponent and dividing by the product of the increased exponent and a, the coefficient of x.*

For example $\int(1+4x)^{\frac{1}{2}}dx=\frac{(1+4x)^{\frac{1}{2}+1}}{4(\frac{1}{2}+1)}=\frac{1}{6}(1+4x)^{\frac{3}{2}}$.

11·32. $\int\frac{dx}{ax+b}$. Here we try $\log_{e}(ax+b)$ which when differentiated gives $\frac{1}{ax+b}\times a$ *i.e.* $\frac{d}{dx}\frac{\log(ax+b)}{a}$

$$=\frac{1}{ax+b}$$ ✓

Hence $\int\mathbf{\frac{dx}{ax+b}=\frac{1}{a}\log(ax+b)}$.

For example, $\int\frac{dx}{7x-9}=\frac{1}{7}\log(7x-9)$.

11·33. $\int\sin(ax+b)\,dx$.

It suggests a trial differentiation of $\cos(ax+b)$, we have $\frac{d}{dx}\cos(ax+b)=-a\sin(ax+b)$

$$i.e.\ \frac{d}{dx}\frac{\cos(ax+b)}{-a}=\sin(ax+b)$$

Hence $\int\mathbf{\sin}(ax+b)dx=-\frac{\cos(ax+b)}{a}$.

Similarly $\int\cos(ax+b)\,dx=\frac{\sin(ax+b)}{a}$, $\int e^{ax+b}dx=\frac{1}{a}e^{ax+b}$

$$\int\sec^{2}(ax+b)\,dx=\frac{\tan(ax+b)}{a}$$

$$\int\text{cosec}^{2}(ax+b)dx=-\frac{\cot(ax+b}{a}$$

In general if $\int f(x)dx=\phi(x)$

then $\int \mathbf{f(ax+b)dx}=\frac{1}{a}\phi(\mathbf{ax+b})$ II

For example, $\int \cos\,(5x+2)dx=\frac{1}{5}\sin\,(5x+2)$,
$\int e^{-3x+2}dx=-\frac{1}{3}e^{-3x+2}$

$$\int \sin\,(9-7x)dx=-\frac{\cos(9-7x)}{-7}=\frac{1}{7}\cos\,(9-7x),$$

$$\int \operatorname{cosec}^2\,(11-9x)=-\frac{\cot\,(11-9x)}{-9}=\frac{1}{9}\cot\,(11-9x)$$

Exercise XXXIII

Integrate the following :—

1. $(x-1)^2$, $\sqrt{1-x}$, $\frac{1}{x-1}$, $\frac{1}{(1+x)^3}$, $\frac{1}{(1-x)^4}$.

$\frac{1.}{\sqrt{1-x}}$

2. $\frac{1}{2x+1}$(P.U.), $\frac{1}{2-x}$, $\frac{1}{(2-x)^3}$, $\frac{1}{(4+5x)^2}$, $\sqrt{4-5x}$.

3. $\sqrt{4-\frac{1}{2}x}$, $(3-2x)^{\frac{3}{3}}$; $(3-4x)^5$ $(2x+3)^{\frac{4}{5}}$; $\sqrt{2x+5}$.

4. e^{5x}, e^{-2x}, e^{7x+3}, e^{ax+b}. 5. 2^x, b^x, x^b, c^{2x}.

6. $\cos 3x$, $\cos 3x+\sin 2x$, $\sec^2\frac{1}{3}\theta$, $\sin\frac{1}{5}(3-2x)$.

7. $\sin\theta\cos\theta$, $\cos^2\theta-\sin^2\theta$, $\sin\,(4-x)$, $\sec^2(nx+a)$.

⑧ $\frac{1}{\sin^2 2x}$, $\frac{\sin x}{\cos^2 x}$, $\frac{\cos x}{\sin^2 x}$, $\frac{\sin\,(3-4x)}{\cos^2\,(3-4x)}$.

Evaluate :

9. (*i*) $\int (\operatorname{cosec}\theta-\cot\theta)\operatorname{cosec}\theta\, d\theta$.

(*ii*) $\int (\tan x+\sec x)\sec x\, dx$.

10. (*i*) $\int \frac{\cos 2x-2\cos^2 x}{\sin^2 x}dx$ (*ii*) $\int \frac{\cos 2x+2\sin^2 x}{\cos^2 x}dx$

(P.U.)

11·4. Constant of Integration. If $\frac{d}{dx}\phi(x)=f(x)$,

then also $\frac{d}{dx}[\phi(x)+c]=f(x) \therefore \int f(x)\,dx=\phi(x)+c.$

The constant c may be give any value, It is called the *constant of integration* or an *arbitrary constant*

The constant is some thing which is usually not written down but it is always supposed to exist. *conversely, if a function has two integrals, they differ by a constant.*

Suppose $\phi_1(x)$, $\phi_2(x)$ are the integrals of $f(x)$,

$$\text{then } \frac{d}{dx}\phi_1(x)=\frac{d}{dx}\phi_2(x)=f(x)$$

$$\therefore \frac{d}{dx}[\phi_1(x)-\phi_2(x)]=\frac{d}{dx}\phi_1(x)-\frac{d}{dx}\phi_2(x)=f(x)-f(x)=0$$

$\therefore \phi_1(x)-\phi_2(x)=c$, a constant *i.e.* $\phi_1(x)=\phi_2(x)+c$.

Thus a function has an infinite number of integrals related by the simple fact that they differ by a constant. Unless a definite value of c is obtained by some known facts, the integrals are known as *indefinite integrals*.

Examples

Ex. 1. If $\frac{dy}{dx}=3x^2$, find y having given that when $x=0$, $y=7$.

Since $dy/dx=3x^2$, $\therefore y=x^3+c$

Putting $x=0$ and $y=7$, we get $c=7$ $\therefore y=x^3+7$.

Ex. 2. If $\frac{dy}{dx}=1-7x^4$, find y given that $y=\frac{3}{5}$ when $x=1$.

Here $y=x-\frac{7}{5}x^5+c$

Putting $x=1$, $y=\frac{3}{5}$, we have $\frac{3}{5}=1-\frac{7}{5}+c \therefore c=1$.

Hence $y=1+x-\frac{7}{5}x^5$.

Ex. 3. Given the velocity of a body $\frac{ds}{dt}=7+32t$, find the distance s in terms of t, if $s=25$ when $t=-1$.

Integrating we have $s=7t+16t^2+c$

Putting $t=-1$, $s=25$, we get $25=-7+16+c$, *i.e.* $c=16$
Hence $s=16+7t+16t^2$.

11.41. It may happen that intergation of the same function by different methods may give results of different forms, *e. g.*

$$\int\frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x \qquad \text{also} \qquad \int\frac{dx}{\sqrt{1-x^2}}=-\cos^{-1}x,$$

for $1/\sqrt{1-x^2}$ is the derivative of both $\sin^{-1}x$ and $-\cos^{-1}x$. It does not, therefore, follow that $\sin^{-1}x=-\cos^{-1}x$; that this is not true can be seen by putting $x=0$ which gives the absurd result $0=-\frac{1}{2}\pi$. What is really true is that they differ by a constant (Art 11·4). In fact, by Trigonometry, $\sin^{-1}x-(-\cos^{-1}x)$ $=\sin^{-1}x+\cos^{-1}x=\frac{1}{2}\pi$.

This shows that the arbitrary constant should always be added to the integral of a function,

Ex. Point out the fallacy in the following :—

$$\int\sin x\cos x\,dx=\tfrac{1}{2}\int\sin 2x\,dx=-\tfrac{1}{4}\cos 2x$$

Also since $\sin x\cos x=\dfrac{d}{dx}\left(\tfrac{1}{2}\sin^2x\right)$

$$\therefore \int\sin x\cos x\,dx=\tfrac{1}{2}\sin^2x$$

$$\therefore \tfrac{1}{2}\sin^2x=-\tfrac{1}{4}\cos 2x \qquad \text{or } 2\sin^2x+\cos 2x=0$$

$$\therefore 2\sin^2x+1-2\sin^2x=0. \qquad \text{Hence } 1=0.$$

11·5. The Definite Integral. In examples given in Art. 11·4, the constant of integration was determined when a pair of values of x and y was given. We sometimes eliminate the constant instead of determining it.

$$\text{Let} \qquad \int f(x)\,dx=\phi(x)+c.$$

and let $x=a$ and $x=b$ be two given values of x, then the *change in the value of the integral* $\phi(x)+c$ as x changes from a to b is equal to

$$[\phi(b)+c]-[\phi(a)+c]=\phi(b)-\phi(a).$$

and is denoted by the symbol $\displaystyle\int_a^b f(x)\,dx$.

This is briefly written as

$$\int_a^b f(x)\,dx = \Big|\,\phi(x)\,\Big|_a^b = \phi(b) - \phi(a). \qquad \text{(A)}$$

The left member of this equation is called the *definite integral* of $f(x)$ between the limits a and b, a being the lower and b the upper limit.

The right hand member of the equation shows that this is found *by integrating $f(x)$ and then subtracting the value of the integral for the lower limit from that of the upper limits.*

Note. The word 'limit' is used here in a sense different from that in Differential Calculus. It means merely the lower or upper end value of x.

Since the constant of integration disappears, there is no point in writing it at all.

Examples

Ex. 1. $$\int_1^4 x^{\frac{3}{2}}\,dx = \Big|\,\tfrac{2}{5}x^{\frac{5}{2}}\,\Big|_1^4 = \tfrac{2}{5}(32) - \tfrac{2}{5} = \tfrac{62}{5} = 12\tfrac{1}{5}.$$

Ex. 2. $$\int_0^{\frac{1}{2}\pi} \cos x\,dx = \Big|\,\sin x\,\Big|_0^{\frac{1}{2}\pi} = \sin\tfrac{1}{2}\pi - \sin 0 = 1 - 0 = 1$$

Ex. 3. $$\int_0^1 \frac{dx}{\sqrt{5x+3}} = \int_0^1 (5x+3)^{-\frac{1}{2}}\,dx$$

$$= \Big|\,\frac{(5x+3)^{\frac{1}{2}}}{5\times\frac{1}{2}}\,\Big|_0^1 = \tfrac{2}{5}\left[8^{\frac{1}{2}} - 3^{\frac{1}{2}}\right] = \tfrac{2}{5}(2\sqrt{2} - \sqrt{3})$$

Ex. 4. $$\int_a^b \frac{1}{x}\,dx = \Big|\,\log_e x\,\Big|_a^b = \log_e b - \log_e a = \log_e \frac{b}{a}.$$

Ex. 5. $$\int_0^1 \frac{dx}{1+x^2} = \Big|\,\tan^{-1} x\,\Big|_0^1 = \tfrac{1}{4}\pi - 0 = \tfrac{1}{4}\pi$$

Ex. 6. $\int \frac{dx}{x\sqrt{x^2-1}} = \left| \sec^{-1} x \right|_1^2$

$$= \sec^{-1} 2 - \sec^{-1} 1 = \tfrac{1}{3}\pi - 0 = \tfrac{1}{3}\pi.$$

11·51. Properties of the Definite Integral. Two of the properties immediately deducible from equation (A)§ 11·5 are given below :—

I. If *the limits of a definite integral are interchanged, the sign of the integral is changed i.e.,*

$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx. \qquad \text{(B)}.$$

For, by (A), $\int_a^b f(x)\, dx = \phi(b) - \phi(a) = -[\phi(a) - \phi(b)]$

$$= -\int_b^a f(x)\, dx.$$

II. If a, b, c are values of x, then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx \qquad \text{(C)}$$

For, the R. H. S. $= [\phi(c) - \phi(a)] + [\phi(b) - \phi(c)] = \phi(b) - \phi(a)$
$=$ L. H. S.

Exercise XXXIV

1. If $\frac{dv}{dt} = 5$ and $v = 3$ when $t = 0$, find v in terms of t.

2. Find the curve

(*i*) which passes through (1, 2) and has its slope equal to $2x$.

(*ii*) having the slope equal to $\frac{3}{2}\sqrt{x}$ and passing through (0,—4).

Evaluate the following definite integrals :—

3. $\int_3^4 x^3\, dx$, $\int_0^2 (1+t)dt$, $\int_0^1 (1-x)^{99} dx$, $\int_{-1}^2 (1-\tfrac{1}{2}x)^2 dx$.

4. $\int_4^9 \frac{dx}{\sqrt{x}}$, $\int_{1/4}^{7/4} \frac{dx}{\sqrt{(4x+1)}}$, $\int_0^1 \sqrt{4+5x}\,dx$,

$\int_1^3 2\sqrt{4x-3}\ dx$.

5. $\int_4^5 \frac{dt}{t^2}$, $\int_{2a}^{3a} \frac{dt}{(a-t)^2}$, $\int_0^3 \frac{27dx}{(2x+3)^3}$,

$\int_{-1}^1 \frac{9dx}{(5-4x)^3}$.

6. $\int_1^4 \frac{dx}{x^{\frac{3}{2}}}$, $\int_{-3}^0 \frac{dx}{(1-x)\sqrt{1-x}}$, $\int_{a^2}^{b^2} \frac{dx}{x}$,

$\int_0^4 \frac{dx}{(2x+1)}$.

7. $\int_0^{\frac{1}{3}\pi} \sin\ x\ dx$, $\int_0^{\pi} \cos x\,dx$, $\int_0^{\frac{1}{4}\pi} \frac{dx}{\cos^2 x}$,

$\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \frac{dx}{\sin^2 x}$.

8. $\int_0^{\pi} \sin 2x\,dx$, $\int_a^b (\cos^2 x-\sin^2 x)dx$, $\int_0^{\frac{1}{3}\pi} \cos 3x\,dx$.

9. $\int_0^{\frac{1}{4}\pi} \sec x\ \tan x\,dx$, $\int_0^{2\pi} \tan x\,dx$, $\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} \cot x\,dx$.

10. $\int_0^1 \frac{dx}{1+x^2}$, $\int_0^{\infty} \frac{dx}{1+x^2}$, $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$,

$\int_1^{\sqrt{2}} \frac{dx}{x\sqrt{x^2-1}}$.

11. Prove that :

(i) $$\int_0^1 \frac{dx}{1+x^2}=\int_0^1 \frac{dz}{1+z^2}.$$

(ii) $$\int_a^b e^x\,dx=-\int_b^a e^x\,dx.$$

(iii) $$\int_0^{\frac{1}{4}\pi} \sin x\,dx=-\int_{\frac{1}{4}\pi}^0 \sin y\,dy.$$

(iv) $$\int_0^1 \frac{dx}{\sqrt{5x+3}}=\int_0^1 \frac{dx}{\sqrt{8-5x}}.$$

(v) $$\int_0^{\frac{1}{2}\pi} \sin x\,dx=\int_0^{\frac{1}{2}\pi} \cos x\,dx.$$

(vi) $$\int_0^1 x^2\,dx+\int_1^3 x^2\,dx=\int_0^3 x^2\,dx.$$

11·6. Theorems relating to integrals. A few fundamental theorems on integration which can be deduced from the corresponding theorems on differentiation are given below :—

11·61. Theorem I. *The operative symbols* $\frac{d}{dx}$ *and* $\int\ldots dx$ *cancel each other,*

$$\textit{i.e.}\ \frac{d}{dx}\left[\int f(x)dx\right]=f(x) \qquad \ldots\text{I}$$

Let $$\frac{du}{dx}=v, \qquad \ldots\ldots(1)$$

where u and v are functions of x, then by the definition of integration, we have

$$u=\int v\,dx \qquad \ldots.(2)$$

$\therefore$ by (1) and (2) $v=\dfrac{du}{dx}=\dfrac{d}{dx}[u]=\dfrac{d}{dx}\left[\int v\,dx\right]$

Thus if $v=f(x), f(x)=\frac{d}{dx}\left[\int f(x)\,dx\right]$

or $\frac{d}{dx}\int f(x)\,dx=f(x)$

For example, $\frac{d}{dx}\int x^4dx=\frac{d}{dx}\left[\frac{x^5}{5}+c\right]=\frac{1}{5}.5x^4=x^4.$

$$\frac{d}{dx}\int \sec^2x\,dx=\frac{d}{dx}(\tan x+c)=\sec^2x.$$

11·62 Theorem II. *A constant factor may be taken out from under the sign of integration and written before it.*

$$i.e. \int \mathbf{cu\,dx}=\mathbf{c}\int \mathbf{udx} \qquad \ldots\mathbf{II}$$

where u is a function of x and c is a constant.

Differentiating the right hand side member of II, we have

$$\frac{d}{dx}\left[c\int udx\right]=c\frac{d}{dx}\int udx=cu, \text{ by theorem I.}$$

Hence by definition of integration, we have

$$\int cu\,dx=c\int u\,dx$$

For example, $\int 5x^4dx=5\int x^4dx=5\times\frac{1}{5}x^5=x^5$
$\int 3\sec^2x\,dx=3\int \sec^2x\,dx=3\tan x$

11·63. Theorem III. *The integral of the (algebraic) sum of a finite number of functions is the sum of their separate integrals.*

$$i.e. \int (\mathbf{u}+\mathbf{v}-\mathbf{w}+\ldots)\mathbf{dx}=\int \mathbf{udx}+\int \mathbf{v\,dx}-\int \mathbf{wdx}+\ldots\text{III}$$

where $u, v, w,$.........are functions of x.

By Differential Calculus, we have

$$\frac{d}{dx}\left[\int udx+\int vdx-\int wdx+\ldots\ldots\right]$$

$$=\frac{d}{dx}\int udx+\frac{d}{dx}\int vdx-\frac{d}{dx}\int wdx+\ldots\ldots$$

$$=u+v-w+\ldots\ldots\ldots\ldots[\text{by theorem I.}]$$

$\therefore$ by definition of integration (ommitting constant), we get

$$\int (u+v-w+\ldots\ldots)dx = \int u\,dx + \int v\,dx - \int w\,dx + \ldots\ldots$$

Examples

Ex. 1. $\int 5x^4+3x^2+6x+7)dx$

$$= \int 5x^4dx + \int 3x^2dx + \int 6dx + \int 7dx$$

$$= 5.\frac{x^5}{5} + 3.\frac{x^3}{3} + 6.\frac{x^2}{2} + 7x.$$

$$= x^5+x^3+3x^2+7x$$

Ex. 2. $\int (\cos 3x + \sin 2x)\,dx$

$$= \int \cos 3x\,dx + \int \sin 2x dx = \frac{\sin 3x}{3} - \frac{\cos 2x}{2}.$$

Ex. 3. $\displaystyle\int \frac{dx}{\sqrt{x+2}-\sqrt{x+1}} = \int \frac{\sqrt{x+2}+\sqrt{x+1}}{(x+2)-(x+1)}dx$

$$= \int \sqrt{x+2}\,dx - \int \sqrt{x+1}dx = \tfrac{2}{3}(x+2)^{\frac{3}{2}} - \tfrac{2}{3}(x+1)^{\frac{3}{2}}$$

Ex. 4. $\int \sin^2 x dx = \int \frac{1}{2}(1-\cos 2x)\,dx$

$$= \tfrac{1}{2}\int 1dx - \tfrac{1}{2}\int \cos 2x\,dx = \tfrac{1}{2}(x - \tfrac{1}{2}\sin 2x).$$

Ex. 5. $\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx$

$$= \int \sec^2 x\,dx - \int 1dx = \tan x - x.$$

Ex. 6. $\int \sin 4x \sin x\,dx = \int \frac{1}{2}\,[\cos 3x - \cos 5x]dx$

$$= \tfrac{1}{2}\,[\tfrac{1}{3}\sin 3x - \tfrac{1}{5}\sin 5x]$$

Ex. 7. $\int (\sec^2 mx - \tan nx)dx = \int \sec^2 mx dx - \int \tan nx dx$

$$= \frac{\tan mx}{m} - \frac{\log \sec nx}{n} = \frac{1}{m}\tan mx + \tfrac{1}{n}\log \cos nx.$$

Exercise XXXV

Perform the integrations :—

1. (*i*) $\int (x^3-4x^2+2x+1)\,dx$

(*ii*) $\int (3x^2-2x^{-3}+4x^{-\frac{5}{3}})\,dx.$

(*iii*) $\int \frac{t^3-t^2+2t+1}{t^4}\,dt$ (*iv*) $\int \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx.$

2. (*i*) $\int \frac{x\,dx}{\sqrt{x+1}}$, (*ii*) $\int \frac{4x+2}{2x-1}\,dx,$

(*iii*) $\int \frac{dx}{\sqrt{x+5}-\sqrt{x+3}}.$

3. (*i*) $\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}.$ (P. U.)

(*ii*) $\int \cos^2 x\,dx.$ (P.U.) (*iii*) $\int \sin^3 \theta\,d\theta$ (P. U)

(*iv*) $\int \cot^2 x\,dx.$ (P. U.)

(*v*) $\int (\sin x-\cos x)^2\,dx.$ (P. U.)

(*vi*) $\int \frac{dx}{1+\cos x}.$ (P. U.)

(*vii*) $\int \frac{dx}{1+\sin x}.$ (P. U.)

(*viii*) $\int \sqrt{1+\cos x}\,dx.$ (P. U.)

(*ix*) $\int (\sec^2 3x-\tan^2 5x)\,dx.$

4. (*i*) $\int \sin 3x \cos x\,dx.$ (P. U.)

(*ii*) $\int \sin 2x \cos^2 x\,dx.$ (P. U.)

(*iii*) $\int (a+b^x+c^{2x})\,dx.$

5. (*i*) $\int (x^5+5^x)\,dx.$ (*ii*) $\int (a^x+x^a)\,dx.$

(*iii*) $\int \left[\frac{1}{\sqrt{x}}+\sin\tfrac{1}{2}x+e^{\frac{1}{2}x}+\frac{2}{x+2}\right]dx.$

6. Evaluate :—

(*i*) $\int_0^2 x(x-1)(x-2)\,dx$

(*ii*) $\int_{3}^{15} \frac{x\,dx}{\sqrt{x+1}}$. (*iii*) $\int_{4}^{5} \frac{4x+5}{x-3}\,dx$.

(*iv*) $\int_{0}^{\pi} \sqrt{1+\sin x}\,dx$. (*v*) $\int_{1}^{2} \pi y^2\,dx$ if $xy=c^2$.

11·7. Differentials. So far we have used $\frac{dy}{dx}$ as a single symbol. Now we shall give meanings to dy and dx, so that dy/dx will also stand for dy divided by dx.

Let $y=f(x)$.

then $$\frac{dy}{dx}=\underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x}=f'(x).$$

$$\therefore \quad \frac{\delta y}{\delta x}=f'(x)+a \text{ where } a \to 0 \text{ as } \delta x \to 0.$$

$$\therefore \quad \delta y=f'(x)\,\delta x+a.\delta x$$

Now when δx is very small, a is also very small so that $a\delta x$ is still smaller and can be neglected in comparison with $f'(x)\,\delta x$.

$f'(x)\,\delta x$ is the principal value of δy. It is called the *differential* of y and is denoted by dy.

$\therefore \quad dy=f'(x)\,\delta x$ ... ... (1)

If we take $y=f(x)=x$, then $f'(x)=1$

and $dy=1\cdot\delta x$ ... ... (*a*)

Also since $y=x$, $dy=dx$ ... . (*b*)

$\therefore$ from (*a*) and (*b*), we get $dx=\delta x$ ... .. (2)

i. e. the differential of the independent variable is equal to its increment.

Substituting dx for δx in (1), we have

$dy=f'(x)\,dx.$ ... ... (3)

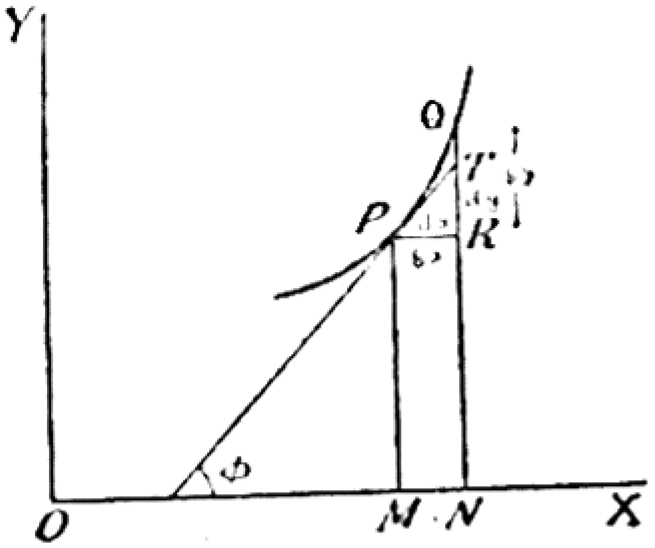
i. e. differential of $y=\frac{dy}{dx}\times$ *differential of* x.

Thus the differential of any function is equal to its derivative multiplied by the differential of the independent variable.

For example, if $y=\sin x$, then $dy=\cos x\, dx$.
If $y=x^2$, then $dy=2x\, dx$;
if $u=a^x$, then $du=a^x \log a.\, dx$:
If $x^2=\sin\theta$, then $2x\, dx=\cos\theta\, d\theta$.

11·71. Geometrical meaning of differential of a function.

Let $P(x, y)$. $Q(x+\delta x, y+\delta y)$ be two points on the curve $y=f(x)$.



Let the tangent at P meet the ordinate of Q in T, then $dx=\delta x=PR$, $RQ=\delta y$. (1)
$RT=PR\ \tan\angle RPT$
$=\tan\phi.\ PR=f'(x)\, dx=dy$.

Hence $RT=dy$ (2)

Therefore dy *is the increment of the ordinate of the tangent line, whereas* δy *is the increment of the ordinate of the curve.*

The differential dy *is thus the amount by which* y *would change while* x *changes by amount* δx *if the point continued to move in the same direction as at P.*

In general it does not continue to move in the same direction and so dy and δy are different. If the arc is sufficiently short, the change in direction is usually small and so dy and δy are nearly equal. In fact $\delta y-dy=TQ$ is a small quantity of higher order than δx.

If s is the distance moved by a body in time t, then since $ds=\dfrac{ds}{dt}.\, dt=v dt$, ds is the distance the body would move in time δt if it continued to move with the speed it had at the time t ; whereas δs is the actual distance moved.

Note. dy, dx have been given definite meanings when standing separately. We shall make use of differentials in the next chapter in reducing certain integrals to standard forms by changing the independent variable.

CHAPTER XII

REDUCTION OF INTEGRALS TO STANDARD FORMS

12·1. If expressions to be integrated are such that it is not easy to guess the functions of which they are the derivatives, it becomes necessary to reduce such expressions to integrable form. There are several devices by which this can be done, a few of them are discussed in this chapter.

12·11. Integration by substitution. Many integrals can be evaluated by introducing a new variable, say z, in place of the original variable x, the two variables being connected by some suitable relation. This procedure is called *integration by substitution.*

Examples

Ex. 1. $\int x\sqrt{a^2+x^2}\,dx.$

The existence of the differential $x\,dx$ suggests the substitution $a^2+x^2=z$, then $2x\,dx=dz$ *i. e.* $x\,dx=\frac{1}{2}dz$,

$$\therefore \quad \int x\sqrt{a^2+x^2}\,dx=\int \tfrac{1}{2}z^{\frac{1}{2}}dx=\tfrac{1}{2}\times\tfrac{2}{3}z^{\frac{3}{2}}=\tfrac{1}{3}(a^2+x^2)^{\frac{3}{2}}.$$

Ex. 2. $\int \frac{x\,dx}{\sqrt{1-x^4}}.$

$x\,dx = \frac{1}{2}dz$

Put $x^2=z$, then $2x\,dx=dz$. or

$$\therefore \quad \int\frac{x\,dx}{\sqrt{1-x^4}}=\tfrac{1}{2}\int\frac{dz}{\sqrt{1-z^2}}=\tfrac{1}{2}\sin^{-1}z=\tfrac{1}{2}\sin^{-1}x^2.$$

Aliter. To free the integrand of radicals, we substitute $x^2=\sin\theta$, then $2x\,dx=\cos\theta\,d\theta$,

$$\therefore \quad \int\frac{x\,dx}{\sqrt{1-x^4}}=\tfrac{1}{2}\int\frac{\cos\theta\,d\theta}{\sqrt{1-\sin^2\theta}}=\tfrac{1}{2}\int\frac{\cos\theta}{\cos\theta}d\theta$$

$$=\tfrac{1}{2}\int 1\,d\theta=\tfrac{1}{2}\theta=\tfrac{1}{2}\sin^{-1}x^2$$

Ex. 3. $\int \sin^5 x \cos x \, dx.$

Put $\sin x = z$, then $\cos x \, dx = dz$

$$\therefore \int \sin^5 x \cos x \, dx = \int z^5 \, dz = \tfrac{1}{6} z^6 = \tfrac{1}{6} \sin^6 x.$$

Ex. 4. $\tan^3 \theta \sec^4 \theta \, d\theta.$

Put $\tan \theta = u$, then $\sec^2 \theta \, d\theta = du$,

$$\therefore \int \tan^3 \theta \sec^4 \theta \, d\theta = \int \tan^3 \theta (\tan^2 \theta + 1) \sec^2 \theta \, d\theta$$
$$= \int u^3 (u^2 + 1) du = \int (u^5 + u^3) du = \tfrac{1}{6} u^6 + \tfrac{1}{4} u^4$$
$$= \tfrac{1}{6} \tan^6 \theta + \tfrac{1}{4} \tan^4 \theta.$$

Ex. 5. $\int \tan^3 \theta \sec^3 \theta \, d\theta.$

Put $\sec \theta = u$, then $\sec \theta \tan \theta \, d\theta = du$

$$\therefore \int \tan^3 \theta \sec^3 \theta \, d\theta = \int \tan^2 \theta \sec^2 \theta \, du = \int (u^2 - 1) u^2 \, du$$
$$= \int (u^4 - u^2) du = \tfrac{1}{5} u^5 - \tfrac{1}{3} u^3 = \tfrac{1}{5} \sec^5 \theta - \tfrac{1}{3} \sec^3 \theta.$$

Note. It is important to note the difference between Ex. 4. and Ex. 5.

Ex. 6. $I = \int \dfrac{e^{\tan^{-1} x}}{1 + x^2} dx.$

Let $\tan^{-1} x = z$, then $\dfrac{dx}{1 + x^2} = dz.$

$$\therefore \quad I = \int e^z \, dz = e^z = e^{\tan^{-1} x}.$$

12·12. When substitution is employed in evaluating a *definite integral*, return to the old variable, is obviated by changing the limits of integration to those of the new variable.

Ex. 1. $\displaystyle\int_0^1 \frac{dx}{(1 + x^2)^{\frac{3}{2}}}$

Put $x = \tan \theta$, then $dx = \sec^2 \theta \, d\theta.$

and $(1 + x)^{\frac{3}{2}} = \sec^3 \theta$, also when $x = 0$, $\theta = 0$
and when $x = 1$, $\tan \theta = 1$ *i.e.* $\theta = \tfrac{1}{4}\pi.$

Hence $$\int_0^1 \frac{dx}{(1+x^2)^{\frac{3}{2}}} = \int_0^{\frac{1}{4}\pi} \frac{\sec^2\theta}{\sec^3\theta}\, d\theta$$

$$= \int_0^{\frac{1}{4}\pi} \cos\theta\, d\theta = \Big| \sin\theta \Big|_0^{\frac{\pi}{4}} = \frac{1}{\sqrt{2}}.$$

Ex. 2. $$\int_0^\infty e^{-x^2}\, x\, dx.$$

Put $x^2 = z$, then $2x\, dx = dz$.

when $x=0$, $z=0$; when $x \to \infty$, $z \to \infty$ so that the limits of integration remain the same in this case.

$$\therefore \int_0^\infty e^{-x^2}\, x\, dx = \tfrac{1}{2} \int_0^\infty e^{-z} dz = \tfrac{1}{2} \Big| -e^{-z} \Big|_0^\infty = \tfrac{1}{2}$$

for when $z \to \infty$, e^{-z} or $1/e^z \to 0$

and when $z=0$, $e^{-z} = 1$.

Ex. 3. $$\int_0^1 \frac{dx}{e^x + e^{-x}} = \int_0^1 \frac{e^x\, dx}{e^{2x} + e^0} = \int_0^1 \frac{e^x\, dx}{1 + e^{2x}}$$

Now put $e^x = z$, then $e^x dx = dz$ and $e^{2x} = z^2$.
Also when $x = 1$, $z = e$

$$\therefore \int_0^1 \frac{e^x\, dx}{1+e^{2x}} = \int_1^e \frac{dz}{1+z^2} = \Big| \tan^{-1} z \Big|_1^e$$

$$= \tan^{-1} e - \tan^{-1} 1 = \tan^{-1} e - \tfrac{1}{4}\pi.$$

Exercise XXXVI

Evaluate the following integrals using the substitution indicated against each :—

1. $\int \frac{2x\, dx}{1+x^4}$ $(x^2 = z)$; $\int \frac{3x^2\, dx}{1+x^6}$ $(x^3 = z)$.

2. $\int x^3 \sin x^4\, dx$ $(x^4 = u)$; $\int e^x \cos e^x\, dx$ $(e^x = t)$

3. $\int x^{n-1} \cos x^n \; dx \; (x^n = t)$;

$\int \cos^2 x \sin^3 x \, dx \; (\cos x = z)$.

4. $\int \cos^5 x \, dx \; (\sin x = t)$; $\int \sin^7 x \, dx \; (\cos x = t)$

Make suitable substitutions to perform the integrations :—

5. $\int \frac{x \, dx}{\sqrt{1-4x^2}}, \; \int \frac{x-2}{\sqrt{2x^2-8x+5}} dx.$

sin x

6. $\int \frac{\cos \sqrt{x} \, dx}{\sqrt{x}}, \quad \int \frac{\cos x \, dx}{1+\sin x}; \quad \int \frac{\cos \theta \, d\theta}{1+\sin^2 \theta}.$

7. $\int \frac{x^{n-1} \, dx}{x^n + a^n}, \quad \int \frac{e^{2x} \, dx}{1+e^{2x}}, \quad \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx.$

8. $\int \frac{dx}{x^2 e^{1/x}}, \quad \int \frac{x^{e-1} + e^{x-1}}{x^e + e^x} dx, \quad \int \frac{(e^x - 1) e^x}{e^x + 1} dx.$

9. $\int \frac{dx}{x \log x}.$

$\int \frac{\tan x \, dx}{\log \sec x}, \quad \int \frac{1}{x} \sin (\log x) dx$ (P. U.)

10. $\int e^{\tan \theta} \sec^2 \theta \; d\theta, \int \frac{dx}{x\sqrt{\log x}}; \quad \int \frac{\log x}{x} dx,$

$\int \frac{dx}{x(1+\log x)}.$ (P. U.)

11. $\int \frac{\tan^{-1} x}{1+x^2} dx \; ; \; \int \frac{dx}{(1+x^2) \tan^{-1} x} \; ; \; \int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx.$

12. $\int \sin^2 x \cos^3 x \, dx \; ; \int \tan \theta \sec^6 \theta \, d\theta$;

$\int \tan^3 \theta \sec^7 \theta \, d($

Evaluate the following integrals :—

13. $\int_2^3 \frac{3x^2-1}{x(x^2-1)}dx$; $\int_0^2 \frac{(x^2+1)\,dx}{4x^3+12x+7}$;

$\int \frac{6x+1}{3\sqrt{x-2}}dx.$

14. $\int_0^{\frac{1}{4}\pi} \sec^2\theta\, d\theta$, $\int_0^{\frac{1}{4}\pi} \tan^2\theta\, d\theta$,

$\int_0^{\frac{1}{4}\pi} \sec^4\theta\, d\theta$, $\int_0^{\frac{1}{4}\pi} \tan^4\theta\, d\theta$.

15. $\int_0^4 \frac{dx}{(1+x)\sqrt{x}}$, $\int_0^{\frac{1}{2}\pi} \sin^6\theta \cos^3\theta\, d\theta$;

$\int_{1/e}^{e} \frac{dx}{x(\log x)^{\frac{1}{3}}}$

16. Show that $\int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi}$ cosec θ cot $\theta\, d\theta$ is meaningless

12.13. Two important results .

I. $\int \frac{f'(x)\,dx}{f(x)} = \log f(x).$

II. $\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}.$ $(n \neq -1).$

These can be immediately deduced by the method of substitution.

Let $\qquad f(x)=u$

then $\qquad f'(x)dx=du.$

$\therefore$ I. $\qquad \int \frac{f'(x)}{f(x)}dx = \int \frac{du}{u} = \log u = \log f(x)$

i. e., the integral of a fraction whose numerator is the diff. coeff. of the denominator is log (denominator).

For example, (*i*) $\int \frac{2ax+b}{ax^2+bx+c}dx = \log(ax^2+bx+c)$

(*ii*) $\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx = \log \sin x.$

(*iii*) $\int \tan x \, dx = -\int \frac{-\sin x}{\cos x} dx = -\log \cos x$

$= \log \sec x.$

And II. $\int [f(x)]^n f'(x)dx = \int u^n \, du = \int \frac{u^{n+1}}{n+1}$

$= \frac{[f(x)]^{n+1}}{n+1} (n \neq -1)$

For example,

(*ii*) $\int (ax^2+bx+c)^n (2ax+b)dx$

$= \frac{(ax^2+bx+c)^{n+1}}{n+1} (n \neq -1)$

(*ii*) $\int \sin^5 x \cos x \, dx = \frac{1}{6} \sin^6 x.$

(*iii*) $\int \tan^n x \sec^2 x \, dx = \frac{\tan^{n+1} x}{n+1}.$

(*iv*) $\int \frac{\sec^2 x \, dx}{\sqrt{\tan x}} = \int (\tan x)^{-\frac{1}{2}} \sec^2 x \, dx$

$= 2(\tan x)^{\frac{1}{2}} = 2\sqrt{\tan x}$

Note. Such integrals can be evaluated directly by proper substitution without referring to the results of Art. 12.13.

Ex. Perform the integrations

$\int \tan^3 x dx, \quad \int \frac{3x+2}{3x^2+4x+7} dx$ (P. U. 1952)

(*i*) We have, $\tan^3 x = \tan x (\sec^2 x - 1) = \tan x \, \text{see}^2 x - \tan x$

$\therefore \int \tan^3 x \, dx = \int \tan x \sec^2 x dx + \int \frac{-\sin x}{\cos x} dx$

$= \frac{1}{2} \tan^2 x + \log \cos x$

(*ii*) We have $3x+2=\frac{1}{2}(6x+4)=\frac{1}{2}\frac{d}{dx}(3x^2+4x+7)$

$$\therefore \int \frac{3x+2}{3x^2+4x+7}dx=\frac{1}{2}\int \frac{6x+4}{3x^2+4x+7}dx$$

$$=\tfrac{1}{2}\log(\text{denominator})$$

$$=\tfrac{1}{2}\log(3x^2+4x+7) \text{ or } \log\sqrt{3x^2+4x+7}.$$

12·2. Integrals of sec x and cosec x.

(*i*) sec x can be written as

$$\frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}=\frac{\sec x\tan x+\sec^2 x}{\sec x+\tan x}$$

Now we put $\sec x+\tan x=z$,

then $(\sec x\tan x+\sec^2 x)\,dx=dz$

$$\therefore \int \sec x\,dx=\int\frac{dz}{z}=\log z=\log(\sec x+\tan x)$$

$$=\log\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}x)$$

[$\because$ by Trigonometry, $\sec x+\tan x=\tan(\frac{1}{4}\pi+\frac{1}{2}x)$.]

(*ii*) Similarly the integral of cosec x can be found by writing

$$\operatorname{cosec} x=\frac{\operatorname{cosec} x(\operatorname{cosec} x-\cot x)}{\operatorname{cosec} x-\cot x}$$

$$=\frac{-\operatorname{cosec} x\cot x+\operatorname{cosec}^2 x}{\operatorname{cosec} x-\cot x}$$

Now if $\operatorname{cosec} x-\cot x=z$,
then $(-\operatorname{cosec} x\cot x+\operatorname{cosec}^2 x)\,dx=dz$

$$\therefore \int \operatorname{cosec} x\,dx=\int\frac{dz}{z}=\log z=\log(\operatorname{cosec} x-\cot x)$$

$$=\log\tan\frac{x}{2}.$$

$$\left[\because \operatorname{cosec} x-\cot x=\frac{1-\cos x}{\sin x}=\frac{2\sin^2\frac{1}{2}x}{2\sin\frac{1}{2}x\cos\frac{1}{2}x}=\tan\frac{x}{2}\right]$$

Aliter. As an alternative method, we proceed as follows :—

(*i*) $\operatorname{cosec} x=\dfrac{1}{\sin x}=\dfrac{1+\tan^2\frac{1}{2}x}{2\tan\frac{1}{2}x}=\dfrac{\frac{1}{2}\sec^2\frac{1}{2}x}{\tan\frac{1}{2}x}$

Now if $\tan \frac{1}{2}x = z$ then $\frac{1}{2}\sec^2\frac{1}{2}x\,dx = dz$

$$\therefore \int \operatorname{cosec} x\,dx = \int \frac{dz}{z} = \log z = \log \tan \tfrac{1}{2}x$$

(*ii*) $\sec x = \operatorname{cosec}(\frac{1}{2}\pi + x) = \operatorname{cosec} u$
where $u = \frac{1}{2}\pi + x$ and $du = dx$

$$\therefore \int \sec x\,dx = \int \operatorname{cosec} u\,du = \log \tan \tfrac{1}{2}u$$
$$= \log \tan(\tfrac{1}{4}\pi + \tfrac{1}{2}x)$$

Hence $\displaystyle\int \mathbf{\sec x\,dx = \log(\sec x + \tan x)}$
$$\mathbf{= \log \tan(\tfrac{1}{4}\pi + \tfrac{1}{2}x).}$$

and $\displaystyle\int \mathbf{\operatorname{cosec} x\,dx = \log(\operatorname{cosec} x - \cot x) = \log \tan \tfrac{1}{2}x}$

12·31. Integrals of

(*i*) $\dfrac{1}{a^2+x^2}$, (*ii*) $\dfrac{1}{\sqrt{a^2-x^2}}$ (*iii*) $\dfrac{1}{x\sqrt{x^2-a^2}}$

(*i*) Put $x = a\tan\theta$, then $dx = a\sec^2\theta\,d\theta$,

$$\therefore \int \frac{dx}{a^2+x^2} = \int \frac{a\sec^2\theta\,d\theta}{a^2+a^2\tan^2\theta}$$
$$= \frac{1}{a}\int \frac{\sec^2\theta}{\sec^2\theta}d\theta = \frac{1}{a}\int 1\,d\theta$$
$$= \frac{1}{a}\theta = \frac{1}{a}\tan^{-1}\frac{x}{a}.$$

This is also equal to $-\dfrac{1}{a}\cot^{-1}\dfrac{x}{a}$ as can be seen by putting $x = a\cot\theta$.

(*ii*) Put $x = a\sin\theta$, then $dx = a\cos\theta\,d\theta$

$$\therefore \int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos\theta\,d\theta}{\sqrt{a^2-a^2\sin^2\theta}} = \int \frac{\cos\theta\,d\theta}{\cos\theta}$$
$$= \int 1\,d\theta = \theta = \sin^{-1}\frac{x}{a}$$

This is also equal to $-\cos^{-1}x/a$ as can be seen by putting $x=a\cos\theta$.

(*iii*) Put $x=a\sec\theta$, then $dx=a\sec\theta\tan\theta\,d\theta$.

$$\therefore \int\frac{dx}{x\sqrt{x^2-a^2}}=\int\frac{a\sec\theta\tan\theta\,d\theta}{a\sec\theta\sqrt{a^2\sec^2\theta-a^2}}$$

$$=\int\frac{\tan\theta\,d\theta}{a\tan\theta}=\frac{1}{a}\int d\theta=\frac{1}{a}\theta=\frac{1}{a}\sec^{-1}\frac{x}{a}.$$

This is also equal to $-\frac{1}{a}\operatorname{cosec}^{-1}\frac{x}{a}$ as can be seen by putting $x=a\operatorname{cosec}\theta$.

*** 12·32. Integrals of**

(*i*) $\dfrac{1}{\sqrt{a^2+x^2}}$ (*ii*) $\sqrt{a^2-x^2}$ (*iii*) $\dfrac{1}{\sqrt{x^2-a^2}}$.

(*i*) This corresponds to (*i*) of Art. 12·31, so that by putting $x=a\tan\theta$, we get

$$\int\frac{dx}{\sqrt{a^2+x^2}}=\int\frac{a\sec^2\theta\,d\theta}{\sqrt{a^2+a^2\tan^2\theta}}=\int\sec\theta\,d\theta$$

$$=\log(\sec\theta+\tan\theta)=\log\frac{x+\sqrt{a^2+x^2}}{a}$$

(*ii*) This corresponds to (*ii*) Art 12·31, putting $x=a\sin\theta$, we get

$$\int\sqrt{a^2-x^2}\,dx=\int\sqrt{a^2-a^2\sin^2\theta}\,a\cos\theta\,d\theta$$

$$=\frac{a^2}{2}\int 2\cos^2\theta\,d\theta=\tfrac{1}{2}a^2\int(1+\cos 2\theta)\,d\theta$$

$$=\tfrac{1}{2}a^2(\theta+\tfrac{1}{2}\sin 2\theta)=\tfrac{1}{2}a^2\theta+\tfrac{1}{2}a\sin\theta.\,a\cos\theta$$

$$=\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{x\sqrt{a^2-x^2}}{2}.$$

(*iii*) This corresponds to (*iii*) of Art 12·31, putting $x=a\sec\theta$, we have

$$\int\frac{dx}{\sqrt{x^2-a^2}}=\int\frac{a\sec\theta\tan\theta\,d\theta}{\sqrt{a^2\sec^2\theta-a^2}}=\int\sec\theta\,d\theta$$

$$=\log\,(\sec\theta+\tan\theta)=\log\frac{x+\sqrt{x^2-a^2}}{a}$$

12·33. Integrals of (*i*) $\dfrac{1}{x^2-a^2}$. (*ii*) $\dfrac{1}{a^2-x^2}$.

We have :

(*i*) $$\frac{1}{x^2-a^2}=\frac{1}{(x-a)\ (x+a)}=\frac{1}{2a}\left[\frac{1}{x-a}-\frac{1}{x+a}\right]$$

$$\therefore\quad\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\left[\int\frac{dx}{x-a}-\int\frac{dx}{x+a}\right]$$

$$=\frac{1}{2a}\log\frac{x-a}{x+a}$$

(*ii*) $$\frac{1}{a^2-x^2}=\frac{1}{(a-x)\ (a+x)}=\frac{1}{2a}\left[\frac{1}{a+x}+\frac{1}{a-x}\right].$$

$$\therefore\quad\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\left[\int\frac{dx}{a+x}+\int\frac{dx}{a-x}\right]$$

$$=\frac{1}{2a}\left[\log\,(a+x)-\log\,(a-x)\right]$$

$$=\frac{1}{2x}\log\frac{a+x}{a-x}.$$

12·34. The results obtained so far are collected in the following table :—

(1) $\displaystyle\int\frac{f'(x)\,dx}{f(x)}=\log f(x).$

(2) $\displaystyle\int[f(x)]^n\,f'(x)\,dx=\frac{[f(x)]^{n+1}}{n+1}\quad(n\neq-1)$

(3) $\displaystyle\int\cot x\,dx=\log\sin x.$

(4) $\displaystyle\int\tan x\,dx=-\log\cos x=\log\sec x.$

(5) $\displaystyle\int\sec x\,dx=\log\,(\sec x+\tan x)=\log\,(\tfrac{1}{4}\pi+\tfrac{1}{2}x).$

(6) $\int \text{cosec}\, x\, dx = \log(\text{cosec}\, x - \cot x) = \log \tan \tfrac{1}{2}x$

(7) $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$ or $-\frac{1}{a} \cot^{-1} \frac{x}{a}$.

(8) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log \frac{a+x}{a-x}$.

(9) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log \frac{x-a}{x+a}$.

(10) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a}$ or $-\cos^{-1} \frac{x}{a}$.

(11) $\int \frac{dx}{x\sqrt{a^2-x^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a}$ or $-\frac{1}{a} \text{cosec}^{-1} \frac{x}{a}$.

(12) $\int \frac{dx}{\sqrt{a^2+x^2}} = \log \frac{x+\sqrt{a^2+x^2}}{a}$.

(13) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log \frac{x+\sqrt{x^2-a^2}}{a}$.

(14) $\int \sqrt{a^2-x^2}\, dx = \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x\sqrt{a^2-x^2}}{2}$.

Exercise XXXVII

Perform the following integrations showing full process :—

1. $\int \sec x\, dx$, $\int \text{cosec}\ x\, dx$, $\int \frac{dx}{x^2+a^2}$, $\int \frac{dx}{x^2-a^2}$. (P. U.)

2. $\int \frac{dy}{y^2+16}$, $\int \frac{dx}{x^2+9}$, $\frac{dx}{4a^2+\frac{1}{4}x^2}$, $\int \frac{dx}{x^2-16}$. (P. U.)

3. $\int \frac{dx}{\sqrt{1-x^2}}$, $\int \frac{dx}{\sqrt{a^2-x^2}}$, $\int \frac{dx}{\sqrt{2a^2-\frac{1}{2}x^2}}$,

$\int \sqrt{a^2-x^2}\ dx$. (P. U.)

4. $\int \frac{4x^2}{1+x^6}dx$, $\int \frac{4x^2\,dx}{\sqrt{1-x^6}}$, $\int \frac{x^2dx}{a^6-x^6}$. (Put $x^3=z$).

5. $\int \frac{x\,dx}{\sqrt{4-x^4}}$, $\int \frac{dx}{x\sqrt{2x^4-4}}$, $\int \frac{2x^3\,dx}{4+x^8}$.

Evaluate the following integrals :—

6. $\int \frac{dx}{\sin x+\cos x}$.

[**Hint.** $\sin x+\cos x=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$].

7. $\int \frac{dx}{a\sin x+b\cos x}$. (P. U.)

[**Hint.** $a\sin x+b\cos x=(a^2+b^2)^{\frac{1}{2}}\sin(x+\phi)$ where $\tan\phi=b/a$].

8. (*i*) $\int \frac{dx}{\sin x\cos x}$. (P. U.)

(*ii*) $\int \frac{\sin^2 x-\cos^2 x}{\sin^2 x\cos^2 x}dx$. (P. U.)

9. (*i*) $\int \frac{(\cos x-\sin x)^2}{\sin x}\,dx$.

(*ii*) $\int \frac{\sin x\cos x\,dx}{\sqrt{1+\sin^2 x}}$. (*iii*) $\int (1+\sec x)^2\,dx$.

10. (*i*) $\int (\tan x+\cot x)\,dx$. (*ii*) $\int (\tan x+\cot x)^2dx$

11. (*i*) $\int (\sec 2x+\operatorname{cosec} 2x)^2\,dx$.

(*ii*) $\int \frac{\sin\frac{1}{3}x}{1+\cos\frac{1}{3}x}\,dx$.

12. (*i*) $\int \frac{dx}{(a^2-x^2)^{\frac{3}{2}}}$. (P. U. 1952).

(*ii*) $\int \frac{x^4\,dx}{4+x^2}$. (P. U. 1950) (*iii*) $\int \frac{dx}{\sqrt{1+12x-9x^2}}$. (P. U.)

(*iv*) $\int \frac{x^2\,dx}{x^2-4}$. (*v*) $\int \frac{dx}{\sqrt{3x-x^2-2}}$.

13. (*i*) $\int \frac{dx}{x^2+2x+10}$. (*ii*) $\int \frac{dx}{x^2-4x+5}$

(*iii*) $\int \frac{dx}{x^2-4x-5}$.

14. (*i*) $\int_0^3 \frac{dy}{9+y^2}$. (*ii*) $\int_0^1 \frac{dz}{\sqrt{4-z^2}}$.

(*iii*) $\int_0^2 \frac{du}{\sqrt{8-2u^2}}$. (*iv*) $\int_0^1 \frac{dx}{x^2-x+1}$.

(*v*) $\int_{-a}^{a} \frac{dx}{\sqrt{a^2-x^2}}$. (*vi*) $\int_0^{\infty} \frac{dx}{(1+x^2)^2}$.

15. (*i*) $\int_0^1 \frac{1-x^2}{1+x^2}\,dx$. (*ii*) $\int_0^{\frac{1}{2}\pi} \frac{1-\cos\theta}{1+\cos\theta}\,d\theta$.

(*iii*) $\int_0^{\frac{1}{2}\pi} \frac{\cos\theta\,d\theta}{1+\sin^2\theta}$. (*iv*) $\int_0^{\frac{1}{4}\pi} \frac{\tan\theta\sec^2\theta}{1+\tan^4\theta}\,d\theta$.

(*v*) $\int_0^{\frac{1}{2}\pi} (\sin\theta-\tan\theta)\,d\theta$.

12·4. Integration by Parts. A powerful method of integration, called *integration by parts*, is derived from the formula for the derivative of the product of two functions.

Let u and v be some functions of x, then

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

Integrating this equation with respect to x, we get

$$uv=\int u\frac{dv}{dx}\,dx+\int v\frac{du}{dx}\,dx.$$

or $$\int u\frac{dv}{dx}\,dx=uv-\int\frac{du}{dx}\,v\,dx. \qquad \text{(A)}$$

so that if we regard u and $\frac{dv}{dx}$ as given functions of x, v is the integral of the second function $\frac{dv}{dx}$ and therefore, the last result (A) may be stated in words as follows ;—

Integral of the product of two functions
=1st function×Integral of 2nd.
—Integral of [Diff of 1st×Integral of 2nd].

Note. Putting $\frac{dv}{dx}=w$ and consequently $v=\int w\,dx$, (A) takes the form $\int uw\,dx=u\int w\,dx-\int\frac{du}{dx}\times\left[\int w\,dx\right]dx.$

The following examples will illustrate the method of using the formula (A).

Examples

Ex. 1. $\int x\cos x\,dx.$

Let us take x as the first function and $\cos x$ as the second.

Then $$\int x\cos x\,dx=x\int\cos x\,dx-\int\left[\frac{d}{dx}(x).\int\cos x\,dx\right\}dx$$

$$=x\sin x-\int 1.\sin x\,dx=x\sin x+\cos x.$$

Note. If we take $\cos x$ as the first function and x as the second,

$$\int \cos x.\, x\, dx = \cos x.\, \frac{x^2}{2} + \int (\sin x.) \frac{x^2}{2} dx$$

But it is no easier to find $\int \sin x \frac{x^2}{2} dx$ than to find the original integral, so that it is necessary to make a trial to find which arrangement will be successful.

Ex. 2. $\int x^2 \log x\, dx.$

Take $\log x$ as the first function and x^2 as the second.

$$\text{Then the integral} = \log x.\frac{x^3}{3} - \int \frac{1}{x}.\ \frac{x^3}{3} dx.$$

$$= \frac{x^3 \log x}{3} - \frac{1}{3} \int x^2 dx.$$

$$= \tfrac{1}{3}x^3 \log x - \tfrac{1}{9}x^3.$$

Unity may be taken as one of the factors for integrating by parts. This is illustrated by the following examples:

Ex. 1. $\int \log x\, dx = \int \log x.\, 1.\, dx = \log x.x - \int \frac{1}{x}.\, x\, dx.$

$$= x \log x - \int 1.\, dx = x \log x - x.$$

$$= x (\log x - 1) = x \log_e (x/e).$$

Ex. 2. $\int \sin^{-1} x\, dx = \int \sin^{-1} x.\, 1\, dx$

$$= x \sin^{-1} x - \int \frac{1}{\sqrt{1-x^2}} x\, dx$$

$$= x \sin^{-1} x + \int \frac{-x}{\sqrt{1-x^2}} dx.$$

$$= x \sin^{-1} x + \sqrt{1-x^2}.$$

12·42 The operation of integration by parts may be repeated several times.

Ex. $\int x^2 \cos x \, dx. = x^2 \sin x - \int 2x \sin x \, dx.$ (1)

and $\int x \sin x \, dx. = x(-\cos x) - \int 1\cdot(-\cos x)dx.$

$$= -x \cos x + \sin x.$$

Substituting this in (1), we have

$$\int x^2 \cos x \, dx = x^2 \sin x + 2x \cos x - 2 \sin x.$$

12·43 $\int \sqrt{a^2 - x^2}. \, dx.$

$$= \int \sqrt{a^2 - x^2}. \; 1 \, dx$$

$$= \sqrt{a^2 - x^2}. \, x - \int \frac{d}{dx} \sqrt{(a^2 - x^2)} x \, dx$$

$$= x\sqrt{a^2 - x^2} + \int \frac{x^2}{\sqrt{a^2 - x^2}} dx$$

$$= x\sqrt{a^2 - x^2} + \int \frac{a^2 - (a^2 - x^2)}{\sqrt{a^2 - x^2}} dx.$$

[Note this step.]

$$= x\sqrt{a^2 - x^2} + \int \frac{a^2}{\sqrt{a^2 - x^2}} dx - \int \sqrt{a^2 - x^2} \, dx.$$

Transposing.

$$2 \int \sqrt{a^2 - x^2} \, dx = x\sqrt{a^2 - x^2} + a^2 \int \frac{dx}{\sqrt{a^2 - x^2}}.$$

$$\therefore \int \sqrt{a^2 - x^2} \, dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}.$$

[Compare this with (*ii*) Art. 12·32].

12.44 $\int e^{ax} \sin bx \, dx.$

$$= \sin bx \, \frac{e^{ax}}{a} - \int \frac{d}{dx}(\sin bx). \frac{e^{ax}}{a} \, dx$$

$$= \frac{e^{ax} \sin bx}{a} - \frac{b}{a} \int e^{ax} \cos bx \, dx \quad \ldots\ldots\ldots \quad (1)$$

Now $\int e^{ax} \cos bx\, dx$

$$= \cos bx \cdot \frac{e^{ax}}{a} - \int \frac{d}{dx} (\cos bx) .\ \frac{e^{ax}}{a}\, dx.$$

$$= \frac{e^{ax} \cos bx}{a} + \frac{b}{a} \int e^{ax} \sin bx\, dx$$

$\therefore$ by substituting in (1), we have

$\therefore \int e^{ax} \sin bx\, dx =$

$$= \frac{e^{ax} \sin bx}{a} - \frac{b}{a} \left\{ \frac{e^{ax} \cos bx}{a} + \frac{b}{a} \int e^{ax} \sin bx\, dx. \right\}$$

$$\therefore \left(1 + \frac{b^2}{a^2}\right) \int e^{ax} \sin bx\, dx = \frac{e^{ax} \sin bx}{a} - \frac{b e^{ax} \cos bx}{a^2}$$

or $(a^2+b^2) \int e^{ax} \sin bx\, dx = e^{ax} (a \sin bx - b \cos bx)$.

Hence $\int e^{ax} \sin bx\, dx = \dfrac{e^{ax}(a \sin bx - b \cos bx)}{a^2+b^2}$

$$= \frac{e^{ax}}{\sqrt{a^2+b^2}} \sin \left(bx - \tan^{-1} \frac{b}{a}\right)$$

Similarly it can be shown that

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2+b^2}$$

$$= \frac{e^{ax}}{\sqrt{a^2+b^2}} \cos \left(bx - \tan^{-1} \frac{b}{a}\right).$$

Exercise XXXVIII.

Integrate the follog expressions :—

1. xe^x, x^2e^x. xe^{ax} x^4e^x.

2. $\log x$, $x \log x$, $x^2 \log x$, $\dfrac{1}{x^3} \log x$, $x^4 \log x$.

3. $x \sin x$ (P.U. 1950) $x^2 \sin x$ (P.U. 1951), $x \sec^2 x$, $x \tan^2 x$.

4. $x \cos 3x$, $x \sin x \cos x$, $x \cos^2 x$, $x \cos nx$, $x^2 \cos ax$.

5. $\tan^{-1}x$, $x \tan^{-1}x$, $x \sin^{-1}x$

6. $e^x \sin x$, $e^x \cos x$, $e^{-2x} \sin 3x$. $e^{2x} \cos x$ (P.U 1953).

7. Integrate $e^x \sin^2 x$ and $e^x \cos^2 x$ and hence show that $\int e^x (\sin^2 x+\cos^2 x)\, dx=e^x$ and account for this result.

8. Evaluate (*i*) $\int e^x \sin x \cos x\, dx$.

(*ii*) $\int e^x \sin x \cos x \cos 2x\, dx$.

9. Calculate (*i*) $\int_0^{\pi} x \sin x\, dx$. (*ii*) $\int_0^{\pi} x^2 \sin x\, dx$.

10. Explain the fallacy in the following :—

$$\int 1.\frac{1}{x} dx=x.\ \frac{1}{x}-\int x\left(-\frac{1}{x^2}\right) dx$$

$$i.e.\ \int \frac{1}{x}\, dx=1+\int \frac{1}{x} dx,\ \therefore\ 0=1.$$

12·5. Integration of Rational Fractions. An algebraic function of the form $f(x)/\phi(x)$ where $f(x)$ and $\phi(x)$ are polynomials in x and $\phi(x)$ is resolvable into factors, can be readily integrated by first splitting the function into partial fractions.

12·51. *When the degree of the numerator is equal to or greater than that of the denominator the numerator is divided by the denominator until it is of lower degree than the denominator.*

Ex. $\int \frac{x^2-x}{x+1} dx$.

By division, we have $\frac{x^2-x}{x+1}=x-2+\frac{2}{x+1}$

$$\therefore\ \int \frac{x^2-x}{x+1} dx=\int \left[x-2+\frac{2}{x+1}\right] dx$$

$$=\tfrac{1}{2}x^2-2x+2\log(x+1).$$

12·52. *When the denominator contains linear factors which are not repeated.*

The expression should be split up into partial fractions by the following rule :—

If $x-a$ is such a factor and $\dfrac{A}{x-a}$ is the corresponding partial fraction then to find A, suppress the factor $x-a$ and put $x=a$ in every other portion of the fraction.

Thus $\dfrac{f(x)}{(x-a)\,F(x)}=\dfrac{A}{x-a}+\dfrac{R(x)}{F(x)}$, where $A=\dfrac{f(a)}{F(a)}$.

The proof of this is left to the student as an exercise

Examples

Ex. 1. Find $\displaystyle\int\frac{x^2+2x+8}{(x-1)(x-2)}\,dx$.

Here the numerator is not of lower degree than the denominator. Therefore first by dividing out we have

$$\frac{x^2+2x+8}{(x-1)(x-2)}=1+\frac{5x+6}{(x-1)(x-2)}$$

Now we break $\dfrac{5x+6}{(x-1)(x-2)}$ into partial fractions by the application of the above rule and get

$$\frac{5x+6}{(x-1)(x-2)}=-\frac{11}{x-1}+\frac{16}{x-2}$$

$$\therefore \text{ the given integral}=\int\left(1+\frac{16}{x-2}-\frac{11}{x-1}\right)dx$$

$$=x+16\log(x-2)-11\log(x-1)$$

Ex. 2. Perform the integration, $\displaystyle\int\frac{x\,dx}{3+4x+x^2}$. (P.U. 1953)

We have

$$\frac{x}{3+4x+x^2}=\frac{x}{(x+1)(x+3)}=\frac{3}{2(x+3)}-\frac{1}{2(x+1)}$$

$$\therefore \text{ the given integral} = \tfrac{1}{2}\int\left(\frac{3}{x+3} - \frac{1}{x+1}\right)dx$$

$$= \tfrac{1}{2}\Big[3 \log (x+3) - \log(x+1)\Big] = \tfrac{1}{2} \log \frac{(x+3)^3}{x+1}$$

Ex. 3. Find $\displaystyle\int \frac{dx}{(1+x^2)(4+x^2)}$

Here $\displaystyle\frac{1}{(1+x^2)(4+x^2)} = \tfrac{1}{3}\left[\frac{1}{1+x^2} - \frac{1}{4+x^2}\right]$

$$\therefore \text{ the given integral} = \tfrac{1}{3}\int\left[\frac{1}{1+x^2} - \frac{1}{4+x^2}\right]dx$$

$$= \tfrac{1}{3} \tan^{-1}x + \tfrac{1}{6} \operatorname{tax}^{-1} \frac{x}{2}.$$

12·53. *When the denominator contains repeated linear factors.*

Split up the expression into partial fraccions as below:—

Ex. 1. Evaluate $\displaystyle\int \frac{dx}{x^2(1-x)}$.

We assume $\displaystyle\frac{1}{x^2(1-x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{1-x}$

$\therefore \quad 1 = Ax(1-x) + B(1-x) + Cx^2$ for all values of x.
If we put $x=1$, we get $C=1$.
It we put $x=0$, we find $B=1$.
To find A, equating co-efficients of x^2,
we get, $0 = -A + C \qquad \therefore \ A = C = 1.$

$$\text{Hence} \quad \int \frac{dx}{x^2(1-x)} = \int\left(\frac{1}{x} + \frac{1}{x^2} + \frac{1}{1-x}\right)dx$$

$$= \log x - \frac{1}{x} - \log (1-x).$$

Ex. 2. Integrate $\displaystyle\frac{x-2}{(1-x^2)(1-x)}$

We have

$$\frac{x-2}{(1-x^2)(1-x)} = \frac{x-2}{(1-x)^2(1+x)} = \frac{A}{1+x} + \frac{B}{1-x} + \frac{C}{(1-x)^2}$$

$$\therefore \quad x-2=A(1-x)^2+B(1+x)(1-x)+C(1+x)$$

Put $x=-1$, 1 and 0, we get $A=-\frac{3}{4}$, $B=-\frac{3}{4}$, $C=-\frac{1}{2}$.

$$\therefore \quad \int\frac{(x-2)dx}{(1-x^2)(1-x)}$$

$$=\int\left[-\frac{3}{4(1+x)}-\frac{3}{4\,(1-x)}-\frac{1}{2(1-x)^2}\right]dx$$

$$=-\tfrac{3}{4}\log\,(1+x)+\tfrac{3}{4}\log\,(1-x)-\tfrac{1}{2}\left(\frac{1}{1-x}\right)$$

$$=\tfrac{3}{4}\log\frac{1-x}{1+x}-\frac{1}{2(1-x)}.$$

12·54. *When the denominator is a quadratic having no linear factor.*

Ex. 1. Find I when $I=\int\frac{x^2+4x+10}{x^2+2x+5}dx$.

Here $\frac{x^2+4x+10}{x^2+2x+5}=1+\frac{2x+5}{x^2+2x+5}$

$$=1+\frac{2x+2}{x^2+2x+5}+\frac{3}{(x+1)^2+4}$$

$$\therefore \quad I=\int\left[1+\frac{2x+2}{x^2+2x+5}+\frac{3}{(x+1)^2+4}\right]dx$$

$$=\ x+\log\,(x^2+2x+5)+\tfrac{3}{2}\tan^{-1}\frac{x+1}{2}.$$

Ex. 2. $I=\int\frac{4x^2+4x-2}{x^3-1}dx$

Assume $\frac{4x^2+4x-2}{x^3-1}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}$

$$\therefore \quad 4x^2+4x-2=A\,(x^2+x+1)+(Bx+C)\,(x-1)$$

Putting $x=1, 0$, we get $A=2$, $C=4$, equating coefficients of x^2, we get $A+B=4$ $\therefore$ $B=2$.

Hence $\frac{4x^2+4x-2}{x^3-1}=\frac{2}{x-1}+\frac{2x+1}{x^2+x+1}+\frac{3}{(x+\frac{1}{2})^2+\frac{3}{4}}$

$$\therefore \quad I = \int \left[\frac{2}{x-1} + \frac{2x+1}{x^2+x+1} + \frac{3}{(x+\frac{1}{2})^2+\frac{3}{4}} \right] dx$$

$$= 2 \log (x-1) + \log (x^2+x+1) + \frac{3}{\sqrt{3/2}} \tan^{-1} \frac{x+\frac{1}{2}}{\sqrt{3/2}}$$

$$= 2 \log (x-1) + \log (x^2+x+1) + 2\sqrt{3} \tan^{-1} (2x+1)/\sqrt{3}.$$

Exercise XXXIX

Integrate the following :—

1. (*i*) $\frac{1}{(x-2)(x-3)}$ (*ii*) $\frac{1}{(x-1)(x-2)}$ (*iii*) $\frac{x}{(x-a)(x-b)}$

(*iv*) $\frac{1}{x^2-16}$ (*v*) $\frac{x^2}{\pi 1-x^6}$ (*vi*) $\frac{2x^2+x}{x-1}$ (P. U.)

2. (*i*) $\frac{x^2}{x-1}$ (*ii*) $\frac{x^3+2x^2+4}{x+2}$ (*iii*) $\frac{x^3+2x+3}{x^2+2}$

3. (*i*) $\frac{1}{x^2+8x-9}$ (*ii*) $\frac{2x+3}{(x+2)(x+3)}$ (*iii*) $\frac{x}{x^2+2x+1}$

(*iv*) $\frac{x^2-5x+6}{x^2+x-12}$ (*v*) $\frac{x^3+6x-8}{x^3-4x}$ (*vi*) $\frac{x^2}{(x-1)(x-2)}$

4. (*i*) $\frac{x}{(x-1)(x+1)(x+2)}$ (*ii*) $\frac{1+x+2x^2}{x(x+1)(2x+1)}$

5. (*i*) $\frac{1}{x^4-1}$ (*ii*) $\frac{x^2}{(x^2+1)(2x^2+1)}$ (*iii*) $\frac{1}{(x^2+a^2)(x^2+b^2)}$

6. (*i*) $\frac{1}{(x-1)^3(x+1)}$ (*ii*) $\frac{x+1}{(x-1)^2(x+2)}$ (*iii*) $\frac{1}{(x^2-1)^3}$

7. (*i*) $\frac{1}{(1-x)\sqrt{x}}$ (P. U.) (*ii*) $\frac{10x-21}{(2x--5)(2x-3)}$ (P.U.)

8. (*i*) $\frac{x+1}{x^2+4x+5}$ (*ii*) $\frac{2x^2+3x+4}{x^2+6x+10}$ (*iii*) $\frac{x^2}{x^2+2x+10}$

Evaluate :—

9. (*i*) $\int \frac{1+\tan^2\theta}{1-\tan^2\theta} d\theta$ (*ii*) $\int \frac{\sec^2\theta \, d\theta}{\tan^3\theta+4\tan\theta}$

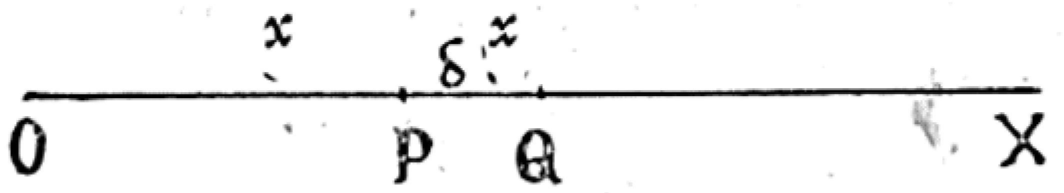
10. (*i*) $\int_4^{\pi/2} \frac{\cos\theta \, d\theta}{\sin\theta+\sin^3\theta}$ (*ii*) $\int_0^{\frac{1}{2}\pi} \frac{\cos x \, dx}{(1+\sin x)(2+\sin x)}$

CHAPTER XIII

APPLICATION OF INTEGRAL CALCULUS

13·1. The integral calculus finds its application in almost all scientific investigations. In this chapter we consider only a few simple cases.

13·2. Application to motion along a straight line. Suppose that a particle moves along a straight line OX,



commencing its motion from the point O, where its velocity is u.

Let it describe a distance x in time t, arriving at P where its velocity is v and acceleration f.

Then by Articles 6·21, 6·3, we have

$$v=\frac{dx}{dt} \qquad \text{and} \qquad f=\frac{dv}{dt} \text{ or } \frac{d^2x}{dt^2}.$$

The particle may be moving,

(*i*) with a constant velocity,

(*ii*) with a variable velocity and constant acceleration.

(*iii*) with a variable velocity and variable acceleration. We illustrate these cases by taking numerical examples.

13·21. Ex. 1. A particle starts from a point at a distance of 15 ft. from A and moves along a straight road with a constant speed of 5 ft. per sec. How far is it from A at the end of 4 secs ?

If s is the distance described in time t, then $\frac{ds}{dt}=5$,

$\therefore$ by integration, we have $\qquad s=5t+c.$

Now since the particle was initially at a distance of 15 ft. from A, *i e.* $s=15$ when $t=0$

$\therefore\ 15=0+c.\qquad$ *i.e.* $c=15.$

Hence $s=5t+15=5\times4+15=35$ ft. when $t=4$ secs., the result is otherwise obvious by simple arithmetic.

Ex 2. A particle starts with a velocity of 6 ft. per sec. and moves along a straight line with a constant acceleration of 3 ft. per sec. per sec. How much distance does it describe in 5 secs. and what is its velocity at the end of this interval ?

If s is the distance described in time t and v be the velocity at the end of this time, we have the acceleration

$$=\frac{dv}{d}=3.$$

$\therefore$ by integration $v=3t+c$

Since the particle starts with a velocity of 6 ft. per sec.

$v=6$ when $t=0\qquad\therefore\ 6=0+c\qquad$ *i.e.* $c=6.$

Hence $\quad v=3t+6 \qquad$ ......(1)

This equation gives the velocity of the particle at the end of any time t. To find the velocity at the end of 5 secs. we put $t=5$ and get

$$v=3\times5+6=21\text{ft. per sec.}$$

Now putting $v=\frac{ds}{dt}$ in (1), we have $\frac{ds}{dt}=3t+6 \quad (2).$

Integrating, (2) we get $\qquad s=\frac{3}{2}t^3+6t+c'$

Since we measure distance and time from the starting point, we have $s=0$ when $t=0\quad\therefore\quad c'=0$

Hence $\quad s=\frac{3}{2}t^2+6t \qquad$ .....(3)

This equation gives the distance described is t sec. Putting $t=5$, we get $\quad s=\frac{3}{2}\times25+6\times5=67\frac{1}{2}$ ft., the distance described in 5 seconds.

Note. The results (1) and (3) are particular cases of the usual general formulæ $v=u+ft$ and $s=ut+\frac{1}{2}ft^2$, u being 6 and f, 3 in the above example.

Ex. 3. A body starts with a velocity of 4 ft. per sec. and moves along a straight line so that its acceleration after a time t is $3t^2+5$; find its velocity and the distance described by it at the end of 3 secs.

Let v be the velocity and s the distance described after t seconds of its motion, then

$$\text{the acceleration} = \frac{dv}{dt} = 3t^2+5 \qquad \ldots(1)$$

Integrating (1) we have, $v=t^3+5t+c$.

Since the body starts with a velocity of 4ft. per sec.,

$\therefore$ when $t=0$, $v=4$, so that $4=0+c$ *i.e.* $c=4$.

Hence $$v=4+5t+t^3 \qquad \ldots\ldots(2)$$

Thus equation (2) gives the velocity v at any time t; to find the velocity at the end of 3 sec., we put $t=3$ and get $v=4+15+27=46$ ft. per sec.

Again from (2), we get $v=\dfrac{ds}{dt}=4+5t+t^3$,

which on integration gives $s=4t+\frac{5}{2}t^2+\frac{1}{4}t^4+c'$.

Since when $t=0$, $s=0$,

$\therefore$ $c'=0$

Hence $$s=4t+\tfrac{5}{2}t^2+\tfrac{1}{4}t^4. \qquad \ldots\ldots(3)$$

$\therefore$ the distance described in 3 sec. is given by

$$s=4\times 3+\tfrac{5}{2}(3)^2+\tfrac{1}{4}(3)^4=12+\tfrac{45}{2}+\tfrac{81}{4}=54\tfrac{3}{4}\text{ ft.}$$

Exercise XL

1. Taking the acceleration due to gravity to be 32 ft. per sec. per sec., find the velocity acquired and the distance fallen in 4 sec. by a body projected vertically down with a velocity of 10 ft. per sec.

2. A body moving in a straight line, starts with a velocity of 15ft. per sec. and its acceleration after t sec. is $12-6t$ ft. per sec. per sec. Find (*i*) the velocity v and distance s from the starting point in terms of time t ; (*ii*) when and where its velocity is zero after the start.

3. If the velocity v of a body be given by the law $v=10+3t$, calculate the distance traversed in the interval $t=1$ to $t=6$.

4. If the speed v ft. per sec. of a body at time t sec. is given by $v=2t+3t^2$, find the distance traversed between $t=2$ and $t=5$. Find also the acceleration at these instants.

5. A body starts with a velocity of 5 ft. per sec. and after t sec. its acceleration is $4-t^2$ ft. per sec. per sec. what is its velocity and what distance does it describe after 3 sec.

6. On a certain curve $\dfrac{d^2y}{dx^2}=6x-2$, if the curve passes through the point $(-1, 2)$ and has at that point the slope 2, find its equation.

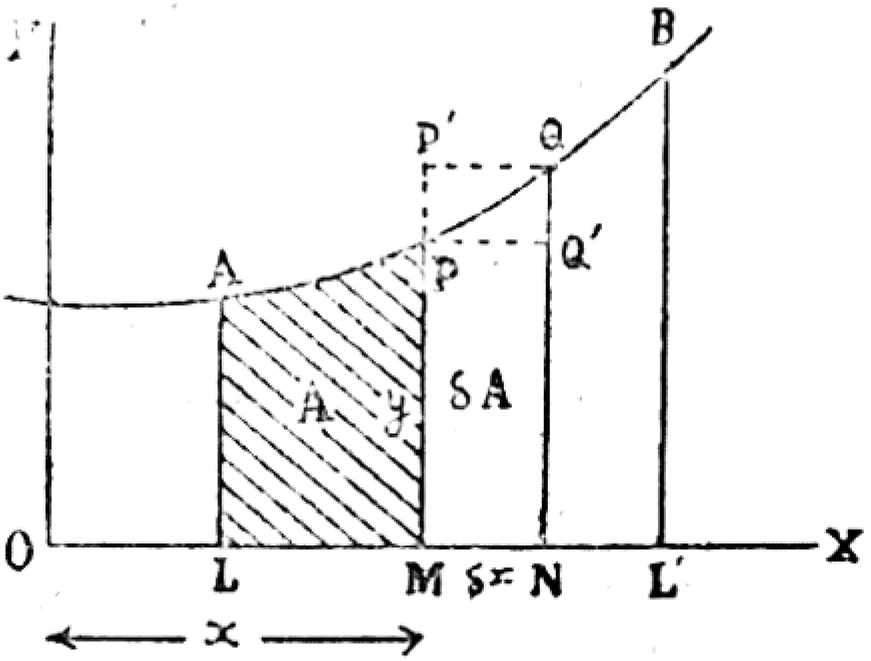
7. The velocity v of a body in terms of t sec. is given by $v=6t^2-22t+12$. Show that the body passes the starting point twice and find the velocity and acceleration on each occasion. Show that it changes direction twice and find the distances from the starting point when this occurs.

13·3. Application to calculation of areas bounded by curves.

A very important application of the integral calculus is to enable us to ascertain the values of areas bounded by curves.

Let APQB be the graph of the curve $y=f(x)$.

Suppose we have to find the area ALL′B, bounded by the curve, the x-axis and the two ordinates LA and L′B and suppose OL$=a$ and OL′$=b$.



Let us denote the area ALMP by A, MP being the ordinate of any point P (x,y) on the curve, so that OM$=x$ and MP$=y$.

As x and y change, the area A must also change. Let δx, δy, δA be the changes in x,y and A respectively, so that MN$=\delta x$, NQ$=y+\delta y$ and the area PMNQ$=\delta$A.

Now the area PMNQ obviously lies between the areas of the rectangles MQ and PN, so that

δA lies between $(y+\delta y)\,\delta x$ and $y\delta x$,

i.e. $\frac{\delta A}{\delta x}$ lies between $y+\delta y$ and y.

$\therefore$ in the limit when $\delta x \to 0$. $\frac{dA}{dx}=y$

$$\text{Hence}\quad A=\int y\,dx+C \text{ or} \int f(x)dx+C=\phi(x)+C \quad \text{......(1)}$$

where $\phi(x)$ is the integral of y or $f(x)$ with respect to x.

When the ordinate MP coincides with LA *i.e.* when $x=a$, A$=0$.

$$\therefore \qquad 0=\phi(a)+C \qquad \text{........(2)}$$

When MP coincides with L′B *i.e.* when $x=b$, let the area be denoted by A′,

$$\text{then} \qquad A'=\phi(b)+C \qquad \text{........(3)}$$

Subtracting (2) from (3), we have

$$A'=\phi(b)-\phi(a)=\int_a^b f(x)\,dx \qquad \text{..........(4)}$$

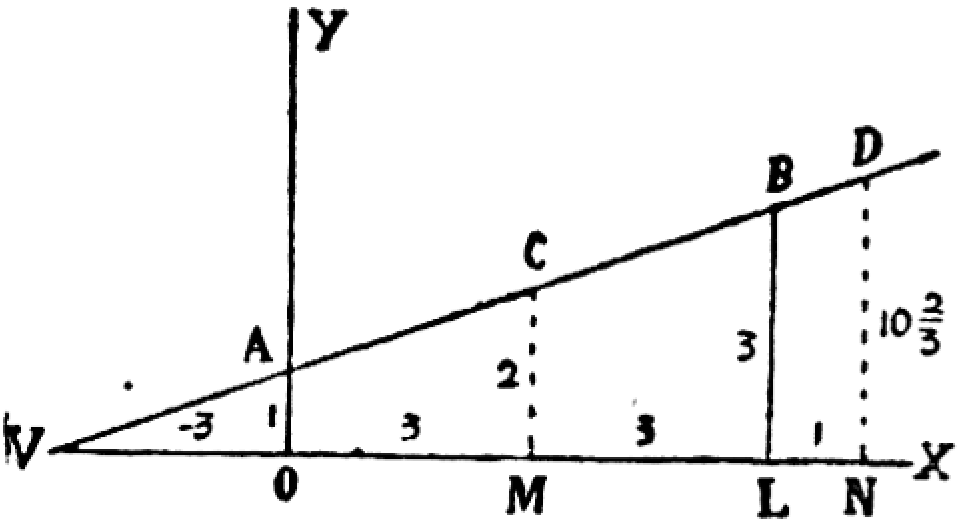
13.31. The length MN$=\delta x$ being very small, the the element PMNQ of the area may be regarded as a rectangle of area $y\delta x$ or $f(x)\,\delta x$. The required area ALL′B from $x=a$ to $x=b$ can be divided into an infinite number of rectangles of which $y\delta x$ is a type. The area is the limit of the sum of areas like $y\delta x$, the summation extending from $x=a$ to $x=b$ and and the limit being taken as the number of rectangles increases beyond any number, however large, so that $\delta x \to 0$. This limit is denoted by the integral

$$\int_a^b y\,dx \quad \text{or} \quad \int_a^b f(x)dx.$$

The integral sign is the elongated form of the letter S and was first used by German mathematician Leibnitz, about 1675.

Ex. *Find the area bounded by the line $y=\frac{1}{3}x+1$, the x-axis and the ordinates* (*i*) $x=0$ and $x=6$ (*ii*) $x=3$ *and* $x=7$.

Let ABCD be the line $y=\frac{1}{3}x+1$. Putting $x=0$, 6, 3, 7, the corresponding values of y are $OA=1$, $LB=3$ $MC=2$ and $ND=3\frac{1}{3}$.



(*i*) Area AOLB$=\displaystyle\int_0^6 (\tfrac{1}{3}x+1)\,dx = \left|\tfrac{1}{6}x^2+x\right|_0^6 = 12$

This is otherwise obvious, for, by geometry, we have area AOLB$=\frac{1}{2}$(OA+LB)OL$=\frac{1}{2}(1+3)\times 6=12$.
(*ii*) In this case the limits are from 3 to 7 and hence

the area CMND$=\left|\tfrac{1}{6}x^2+x\right|_3^7=\tfrac{1}{6}(49-9)+(7-3)=6\tfrac{2}{3}+4=10\tfrac{2}{3}$

Geometrically area CMND$=\frac{1}{2}$ (MC+ND)MN
$=\frac{1}{2}(2+\frac{10}{3})\times 4=10\frac{2}{3}$

13·41. *The area bounded by parabola $y^2=4ax$ and the ordinate $x=h$.*

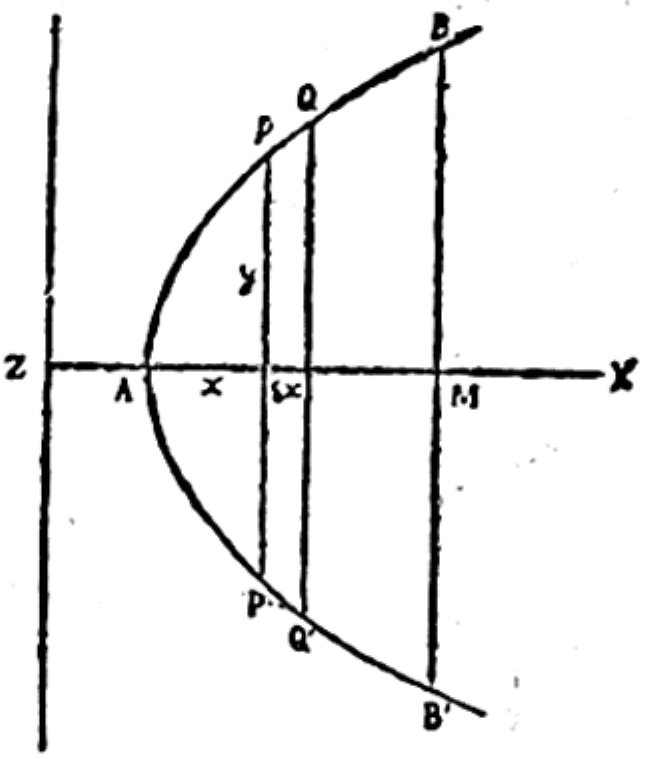
we are required to find the area BAB′ which is twice the area BAM where AM$=h$, $MB^2=4ah$ or $MB=2\sqrt{ah}$.

$\therefore$ the required area

$$=2\int_0^h y\,dx=2\int_0^h 2a^{\frac{1}{2}}\,x^{\frac{1}{2}}dx$$

$$=4a^{\frac{1}{2}}\left|\tfrac{2}{3}x^{\frac{3}{2}}\right|_0^h=\tfrac{8}{3}a^{\frac{1}{2}}\,h^{\frac{3}{2}}$$

$$=\tfrac{8}{3}h\sqrt{ah}.$$



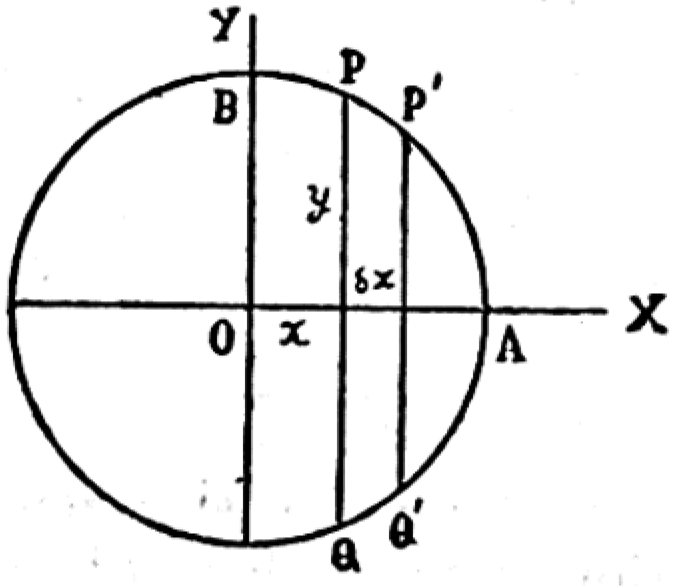
Note. Since $\frac{8}{3}h\sqrt{ah}=\frac{2}{3}h\times4\sqrt{ah}=\frac{2}{3}AM.2MB=\frac{2}{3}AM{\cdot}B'B$

$\therefore$ the area is two thirds of the rectangle formed by the limiting ordinate and the abscissa.

13·42. The area of a circle of radius a. Taking the centre of the circle as the origin and any two perpendicular diameters as the co-ordinate axes, the equation of the circle is

$$x^2+y^2=a^2 \quad \ldots\ \ldots\ \ldots \quad (1)$$

The area of the circle is four times the area of the quadrant AOB.



$\therefore$ the required area

$$=4\int_0^a y\,dx=4\int_0^a\sqrt{a^2-x^2}\,dx$$

$$=4\left|\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right|_0^a$$ [see Art. 10 Ex. 12·32 (*ii*) P. 223. Also see Q. 10 Ex. XLI].

$$=2a^2\sin^{-1}1=2a^2.\frac{\pi}{2}=\pi a^2.$$

Thus the area of a circle of radius a is πa^2 a result already known to the students.

Exercise XLI.

1. Find by integration the area of

(*i*) the triangle formed by the line $y=\frac{1}{3}x+1$, the x-axis and the ordinate $x=6$.

[Triangle VLB fig. Art 13·31]

(*ii*) Of the trapezoid formed by the line $y=2x+3$, the x-axis and the ordinates $x=2$ and $x=10$.

2. Find the area between the curve $y=6x-x^2$, the x-axis and the ordinates $x=2$ and $x=3$.

3. Find the area of the curve $y=x^2+x-5$ between the limits $x=0$ and $x=6$.

4. Find the area between the curve $y=x^3-3x^2+2x$, the x-axis, and the ordinates $x=0$ and $x=1$.

5. Find the area of the parabola $y=2c\sqrt{x}$ between $x=0$ and $x=c$ and show that it is two-thirds of the rectangle formed by the limiting ordinate and its abscissa.

6. Find the area bounded by the hyperbola $xy=c^2$, the x-axis, and the ordinates $x=b$ and $x=a (a>b)$.

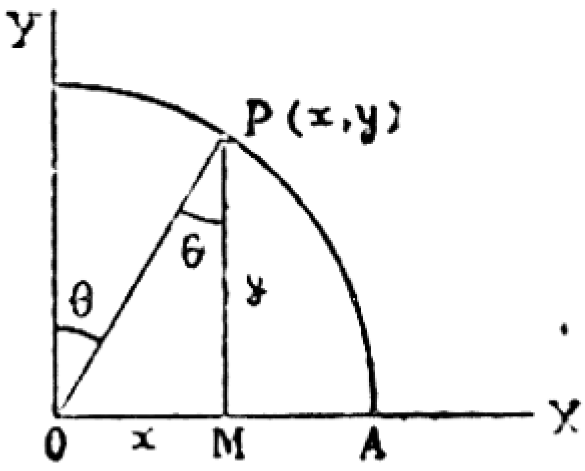
7. Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

8. Find the area of the curve (*i*) $a \sin x$ (*ii*) $a \sin^2 x$ between $x=0$ to $x=\pi$.

9. Find the area between the limits $x=0$ to $x=a$ of the curve (*i*) $y=b/(x+a)$ (*iii*) e^x (*ii*) ce^{-x}

10. Show geometrically that

$$\int\sqrt{a^2-x^2}\,dx=\tfrac{1}{2}x\sqrt{a^2-x^2}+\tfrac{1}{2}a^2\sin^{-1}\frac{x}{a}.$$



Let P(x, y) be any point on the circle $x^2+y^2=a^2$ referred to two perp. radii OA, OB as axes and let $\angle POB=\theta$ then $OM=x$, $MP=y=\sqrt{a^2-x^2}$ and

$$\sin\theta=\sin\angle OPM=\frac{OM}{OP}=\frac{x}{a}$$

i.e. $\theta=\sin^{-1}\frac{x}{a}$.

Now $\int y\,dx=\int\sqrt{a^2-x^2}\,dx=$ area OMPB

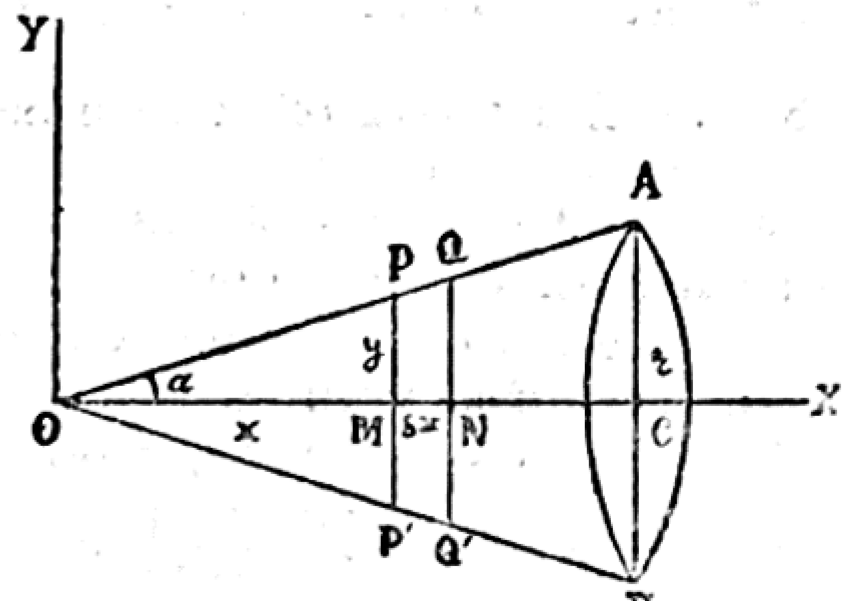
$$=\triangle OMP+\text{ sector } POB$$

$$=\tfrac{1}{2}xy+\tfrac{1}{2}a^2\theta=\tfrac{1}{2}x\sqrt{a^2-x^2}+\tfrac{1}{2}a^2\sin^{-1}\frac{x}{a}.$$

13·5. Application to Calculation of volumes of revolution. Suppose the curve $y=f(x)$ [See Fig., Art 13·3] revolves about the x-axis. Then the area PMNQ forms a disc of volume approximately equal to $\pi y^2\delta x$.

Taking δx to be infinitesimally small and summing up such elements in the form of discs, we obtain the volume formed by the revolution about the x-axis, of the area ALL′B$=\int_a^b \pi\, y^2\, dx$.

13·51. The volume of a right circular cone. Let a straight line OA revolve about another line OC so as always to make a constant angle α with it. Then a cone is formed with O as the vertex and OC as the axis.



Let the vertex O be taken as the origin, OC as the x-axis and OY, a straight line perp. to OX in the plane AOC, as the y-axis.

If $\angle COA=\alpha$, the equation of the line OA is

$$y=x\tan\alpha \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

Let $P(x, y)$ and $Q(x+\delta x, y+\delta y)$ be two points on the line OA; MP, NQ being the ordinates of the points. Then as the area PMNQ revolves about OC, it forms a disc of volume approximately equal to $\pi y^2\delta x=\pi x^2\tan^2\alpha\,\delta x$, [by (1)]

If OC $= h$, CA$=r$, then $r=h\tan\alpha$ and the required volume.

$$=\int_0^h \pi y^2 dx = \int_0^h \pi x^2 \tan^2 \alpha \, dx = \pi \tan^2\alpha \left| \tfrac{1}{3}x^3 \right|_0^h$$

$$=\tfrac{1}{3}\pi h^3 \tan^2\alpha = \tfrac{1}{3}\pi\ (h\tan\alpha)^2 h\ = \tfrac{1}{3}\pi r^2 h.$$

Volume of the cone is equal to one-third the volume of the cylinder of the same height and base radius.

Ex. Find the volume of the frustum of the cone formed by the revolution about the x-axis, of the line $y=\frac{1}{3}x+1$ included between the y-axis and the ordinate $x=6$. [See Fig. Art 13·31].

The required volume$=\displaystyle\int_0^6 \pi\, y^2 dx = \pi \int_0^6 (\tfrac{1}{3}x+1)^2\, dx$

$$=\pi \int_0^6 (\tfrac{1}{9}x^2+\tfrac{2}{3}x+1)\, dx$$

$$=\pi \left[\tfrac{1}{27}x^3+\tfrac{1}{3}x^2+x\right]_0^6$$

$$=\pi\,(8+12+6)=26\pi=81{\cdot}78.$$

13·52. The volume of a sphere of radius a. Let the sphere be formed by the revolution of the circle

$$x^2+y^2=a^2 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (i)$$

about the x-axis [See Fig. Art 13·42].

Here the required volume$=\displaystyle\int_{-a}^{a} \pi y^2\, dx = \int_{-a}^{a} \pi(a^2-x^2)\, dx$

$$=\pi \left| a^2x-\tfrac{1}{3}x^3 \right|_{-a}^{a}$$

$$=\pi\left[\tfrac{2}{3}a^3-(-\tfrac{2}{3}a^3)\right]=\tfrac{4}{3}\pi a^3,$$

a result already known to the student.

13·53. The volume of a paraboloid. Let a paraboloid be formed by the revolution of the parabola

$y^2=4ax$ about its axis, the parabola being bounded by the ordinate $x=h$. [See Fig. Art 13·41].

$$\text{The required volume } = \int_0^h \pi y^2 dx = \pi \int_0^h 4ax\, dx$$

$$= 2\pi a \left| x^2 \right|_0^h = 2\pi a h^2.$$

Note. Since $2\pi a h^2 = \frac{1}{2}\pi h(4ah) = \frac{1}{2}\pi \text{AM}.\text{MB}^2$,

$\therefore$ the volume of the paraboloid is equal to half the volume of the cylinder whose height is AM and whose base is the circle with diameter B′B.

Exercise XLII.

1. Find the volumes generated by the revolution about x-axis of the curves :—

(*i*) $y=a \sin x$, one arch.
(*ii*) $y=a \cos x$ between $x=0$ and $x=\frac{1}{2}\pi$.
(*iii*) $y=a \tan x$ between $x=0$ and $x=\frac{1}{4}\pi$.
(*iv*) the parabola $y^2=4ax$ bounded by the latus-rectum.
(*v*) the hyperbola $xy=c^2$ bounded by $x=b$ and $x=a$ $(a>b)$.

2. Find the volume of the ellipsoid formed by the revolution of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (*i*) about the major axis, (*ii*) about the minor axis.

3. Find the volume generated by the curve $y^2-x^2=1$ as it revolves about the x-axis, between $x=0$ and $x=4$.

4. Find the the volume generated by the curve $a^2y^2=x^3(2a-x)$ about the x-axis included between its points of intersection with the x-axis.

PUNJAB UNIVERSIY PAPERS.

1951 (April)

1. (*a*) Starting from first principles, obtain the differential coefficients of

(*i*) $\log x$.
(*ii*) $1/\sqrt{x}$.

(*b*) Differentiate with respect to x :

(*i*) $\cos^{-1}(\tan x^2)$.
(*ii*) $\tan^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}}$.

2. (*a*) Find the *n*th differerntial coefficients of

(*i*) $\sin\sqrt{2x}$.
(*ii*) $\cos^2 x$.

(*b*) If $f(x)=x^3+3x^2+15x-24$, find the value of $f(\frac{11}{10})$ by applying Taylor's Theorem.

3. (*a*) Obtain the equation of the tangent at any point to the curve $y=f(x)$.

(*b*) Prove that $\dfrac{x}{a}+\dfrac{y}{b}=1$ touches the curve $y=be^{-\frac{x}{a}}$ at the point where the curve crosses the axis of y.

4. Integrate any three of the following with respect to x:

(*i*) $\int\sqrt{1+\cos x}\,dx$.
(*ii*) $\int\operatorname{cosec} x\,dx$.
(*iii*) $\int\cot^2 x\,dx$,
(*iv*) $\int\dfrac{x\,dx}{a^4+x^4}$.
(*v*) $\int x^2\sin x\,dx$,

5. (*a*) If $y=\epsilon^{ax}\cos(bx+c)$, show that

$$y_n=(a^2+b)^{\frac{1}{2}n}\epsilon^{ax}\cos(bx+c+\tan^{-1}b/a).$$

(*b*) Find the abscissa of the point on the curve $ay^2=x^3$, the normal at which cuts off equal intercepts from the co-ordinate axes.

6. (*a*) Applying Macluarin's Theorem prove that.

$$e^{\sin x}=1+x+\frac{x^2}{1\cdot 2}-\frac{3x^4}{1\cdot 2\,3\cdot 4}+ \ldots\ldots\ldots\ldots\ldots\ldots$$

(*b*) The graph of $y=\dfrac{ax+b}{(x-1)(x-4)}$ has a turning point at P(2,−1). Determine the values of *a* and *b* and show that *y* is max. at P.

1951 (Sept).

1. (*a*) Define differential coefficient and from your definition obtain the differential coefficient of tan *x*.

(*b*) Differentiate with respect to *x*

(*i*) $a^{2x}\; e^{\sqrt{x^2+1}}$.

(*ii*) $\sin^{-1}\dfrac{2x}{1+x^2}$.

2. Find the *n*th differential coefficient of

(*i*) $\sin 2x\ \sin 3x$. (*ii*) $\log(ax+b)$.

(*b*) If $y=e^{ax}\sin bx$, show that

$$y_2-2ay_1+(a^2+b^2)\,y=0.$$

3. (*a*) Show how to find the max. and min. values of a function.

(*b*) Examine the following functions for max. and min. values.

(*i*) $a\sin x+b\cos x$.

(*ii*) $x^5-5x^4+5x^3-1$.

4. Integrate any three of the following.

(*i*) $\displaystyle\int\frac{dx}{\sin x\,\cos x}$. (*ii*) $\int x\cos x^2\,dx$.

(*iii*) $\int \sin^3 x\,dx$.

(*iv*) $\int \sin x\sec^2 x\,dx$.

(*v*) $\dfrac{dx}{\sqrt{9-16x^2}}$.

5. (*a*) Show that in the exponential curve $y=\varepsilon^{\frac{x}{a}}$, the subtangent is of constant length and subnormal varies as the square of the ordinate.

(*b*) If $\sin y=x \sin (a+y)$, prove that

$$\frac{dy}{dx}=\frac{\sin^2(a+y}{\sin\ a}.$$

6. (*a*) State Taylor's Theorem and apply it to expand $\log\sin(x+h)$ and hence find the value of $\log\sin 31^\circ$.

(*b*) Expand by Maclaurin's Theorem $\sin^{-1}x$ in powers of x.

1952 (April).

1. (*a*) Differentiate **ab initio**

(*i*) $\sqrt{ax+b}$. (*ii*) $\sin^{-1}x$.

(*b*) Differentiate with respect to x

(*i*) $x^{\tan x}+(\tan ax)^{ax}$.

(*ii*) $\sqrt{\sin\sqrt{x}}$.

2. (*a*) Find the nth differential coefficient of

(*i*) $\log (ax+b)$.

(*ii*) $\cos^3 x$.

(*b*) Use Maclaurin's Theorem to prove that

$$\tan^{-1}(1+x)=\frac{\pi}{4}+\frac{x}{2}-\frac{x^2}{4}+\frac{x^3}{12}+\dots\dots\dots\dots\dots$$

3. Show that $f(x)$ is max. at $x=a$ if $f'(a)=0$ and $f''(a)$ is negative.

(*b*) Examine for max. or min. the function

$3\sin^2 x+4\cos^2 x$.

4. Perform the following integration with respect to x.

(*a*) $\displaystyle\int\frac{dx}{x^2-a^2}$ or $\displaystyle\int\frac{3x+2}{\sqrt{3x^2+4x+7}}dx$.

(*b*) $\int \tan^3 x\, dx$ or $\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$.

(*c*) $\int \frac{dx}{(a^2-x^2)^{3/2}}$.

5. (*a*) Find the condition that the conics $lx^2+my^2=1$ and $l'x^2+m'y^2=1$ may cut orthogonally.

(*b*) Show that the triangle of max. area inscribed in a circle of radius a is an equilateral triangle.

6. If $f(x) = x^3-2x+5$, find the value of $f(2{\cdot}001)$ with the help of Taylor's formula.

1952 (Sept.)

1. (*a*) Differentiate with respect to x from first principles, (*i*) $\tan x$. (*ii*) $\log_a x$.

(*b*) Differentiate with respect to x

(*i*) $\sqrt{\frac{x^2-2ax}{bx-cx^3}}$.

(*ii*) if $ax^2+cxy+by^2=1$, find $\frac{dy}{dx}$.

2. (*a*) Find the *n*th derivative with respect to x of $\epsilon^{ax} \sin(bx+c)$.

(*b*) Deduce Maclaurin's formula from Taylor's theorem and state the conditions for the validity of the formula.

3. (*a*) Give the geometrical interpretation of $\frac{dy}{dx}$.

(*b*) Find the angle between the two curves

(*i*) $y=x^3$. (*ii*) $6y=7-x^2$.

(*c*) Find the equation of the normal at 'θ' to the curve $x=a\cos\theta$, $y=b\sin\theta$.

4. Perform the following integration with respect to x.

(*a*) $\int \frac{x\,dx}{(2x^2+3)^3}$ or $\int \frac{dx}{\sqrt{1+12x-9x^2}}$.

(*b*) $\int \sec x \, dx$ or $\int \frac{dx}{9a^2+5x^2}$.

(*c*) $\int \sqrt{a^2-x^2} \, dx$.

5. (*a*) It is required to form equal circular cylindrical closed can of maximum volume from a thin sheet of metal ; find the relation between the height and the radius of the base of the cans so as to obtain the best results.

(*b*) Show that the sum of the intercepts of the tangent at any point to the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ on the axes is constant.

6. (*a*) Expand $\log \frac{\tan x}{x}$.

(*b*) Find $\frac{dy}{dx}$ if (*i*) $y=\cos^{-1}\frac{a+b\cos x}{b+a\cos x}$.

(*ii*) $y=(\sin x)^{\tan x}+(\tan x)^{\sin x}$

1953 (April)

1. (*a*) Find from definition the differential coefficients of the following with respect to the variable x,

(*i*) x^n (*ii*) n^x (*iii*) $\sqrt{\sin x}$ where n is a positive integer.

(*b*) Differentiate with regard to x

$\cos(\sin\sqrt{ax+b})$.

2. (*a*) Differentiate with respect to x

(*i*) $\tan^{-1}(\sin x + \cos x)$.

(*ii*) $\log \frac{a+b\sin x}{b-a\sin x}$.

(*b*) If $y=\frac{\sin^{-1} x}{\sqrt{1-x^2}}$, show that

$(1-x^2)\,y_1-xy=1$.

3. (*a*) (*i*) Find the *n*th differential coefficient with respect to x of $\sin(ax+b)$.

(*ii*) If $y=e^{ax}\sin bx$, show that

$$\frac{d^2y}{dx^2}+(a^2+b^2)y=2a\frac{dy}{dx}.$$

(*b*) Apply Maclaurin's formula to deduce that

$$e^{\sin x}=1+x+\frac{x^2}{1.2}-\frac{3x^4}{1.2.3.4}-\ldots\ldots\ldots$$

4. (*a*) Find the equations of the tangent and the normal to the ellipse $9x^2+25y^2=225$ at the point

$$\left(\frac{5}{2},\frac{3\sqrt{3}}{2}\right).$$

(*b*) Proved that the least perimeter of an isosceles triangle which can be circumscribed to a circle of radius a is $6\sqrt{3}a$.

5. (*a*) Differentiate with respect to x

(*i*) $\dfrac{e^{x^2}\tan^{-1}x}{\sqrt{1+x^2}}$. (*ii*) $\cos^{-1}\dfrac{a-b\sin x}{b+a\sin x}$.

(*b*) If $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$, find $\dfrac{dy}{dx}$.

6. Integrate with respect to x :—

(*a*) $\displaystyle\int\frac{dx}{a^3+x^2}$ or $\displaystyle\int\frac{xdx}{3+4x+x^2}$.

(*b*) $\int\sec x\,dx$ or $\int\cos^3 x\sin x\,dx$.

(*c*) $\int x\sin x\,dx$ or $\int e^{ax}\cos x\,dx$.

1953 (Sept.)

1. (*a*) Differentiate with respect to x ab-initio

(*i*) $(ax+b)^n$ where n is a positive integer

(*ii*) $\sec x$ (*iii*) $\sqrt{ax+b}$

(*b*) Differentiate $\cot(\sin\sqrt{x})$.

2. (*a*) Differentiate with respect to x

(*i*) $\log_a x$ (*ii*) $\text{Sin}^{-1}\dfrac{a\cos x-b\sin x}{b\sin x+a\cos x}$

(*b*) Find $\dfrac{dy}{dx}$ if $(\tan x)^y=(\tan y)^x$.

3. (*a*) Find the *n*th derivative with regard to x of

(*i*) $\sqrt{x+a}$ (*ii*) $e^x \sin x$

(*b*) Obtain the expansion

$$e^x \sec x = 1 + x + \frac{2x^2}{\underline{|2}} + \frac{4x^3}{\underline{|3}} + \ldots$$

4. (*a*) Interpret $\frac{dy}{dx}$ geometrically and find the angle at which $y^2 = 2x$ and $x^2 + y^2 = 8$ intersect.

(*b*) A cone is circumscribed about a sphere of radius a; show that when the volume of the cone is a minimum, its altitude is $4a$ and its semi-vertical angle is $\sin^{-1} \frac{1}{3}$.

5. (*a*) Differentiate with respect to x

(*i*) $\log \frac{a + b \cos x}{b + a \cos x}$ (*ii*) $\log \frac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} - \sqrt{a-x}}$

(*b*) Differentiate $7x^5 - 11x^2$ with respect to $7x^3 - 15x$.

6. Integrate with respect to x

(*a*) $\int \frac{dx}{\sin x + \cos x}$ *or* $\int \frac{x^2\,dx}{\sqrt{9 - x^6}}$

(*b*) $\int \sin^2 x\,dx$ *or* $\int \frac{(3x+2)\,dx}{(x+1)(2x+2)}$

(*c*) $\int x \log x\,dx$ *or* $\int e^x . x^2 dx$

ANSWERS

Ex. I-A. Page 3.

1. 48, 6, 0, −12. **3.** x^3+2x^2+2x ; x^3-4x^2+6x-4
4. $-1, 0, 1, \sqrt{2}$.

Ex. I-D. Page 5.

1. (*i*) $x \geqslant 0$ (*ii*) $x \leqslant 1$ and $x \geqslant 2$ (*iii*) $-1 \leqslant x \leqslant 1$.
2. (*i*) $x=2$ (*ii*) $x=1$ and $x=2$ (*iii*) $x=-3$ and $x=3$ (*iv*) For all values of x which are odd multiples of $\frac{1}{2}\pi$. (*v*) $-1<x<1$.

Ex. I-E Page 5.

4. (*i*) 3. (*ii*) 5. **5.** (*i*) −4. (*ii*) −2.
6. 100, 44, 22. **8.** $6, 6+3h+h^2, 3+h$.

Ex. II-A. Page 12.

2. (*a*) $1\frac{1}{2}$. (*b*) $-\frac{3}{4}$. (*c*) $\frac{5}{8}$.

Ex. II-B. Page 14.

1. $\frac{1}{2}$. **2.** $-\frac{1}{2}$. **3.** $\frac{2}{3}$. **4.** (*i*) 2. (*ii*) $2a$. (*iii*) $3h^2$.
5. (*i*) b/a. (*ii*) a/b. **6.** (*i*) 1. (*ii*) ∞. (*iii*) 0.

Ex. II-C. Page 16.

2. (*i*) $\frac{1}{2}$. (*ii*) $-\frac{3}{2}$..

Ex. II-D. Page 17·

1. $3-h$, 3. **2.** 3. **3.** $2x$.

Ex. III. Page 24.

1. (*i*) $3x^2$. (*ii*) 9. (*iii*) $2ax$. (*iv*) $4\pi x^2$. (*v*) $2\pi r$. (*vi*) $6-2x$. (*vii*) $u+gt$.

2. (*i*) $-\frac{3}{x^2}$. (*ii*) $-\frac{2c}{x^3}$. (*iii*) $-\frac{a}{z^2}$. (*iv*) $-\frac{1}{z^2}-\frac{2}{z^3}$ (*v*) $1-\frac{1}{x^2}$. (*vi*) $1-\frac{3}{(x+2)^2}$. (*vii*) $\frac{\sqrt{5}}{2\sqrt{z}}$. (*viii*) $1-\frac{1}{2}x^{-\frac{3}{2}}$

(*ix*) $\dfrac{1}{2\sqrt{x+c}}$. (*x*) $\dfrac{x}{\sqrt{x^2+c^2}}$.

3. (*i*) $-\dfrac{1}{(x+7)^2}$; $-\dfrac{1}{49}$, $-\dfrac{1}{121}$

(*ii*) $\dfrac{2}{(5-2x)^2}$; $\dfrac{2}{25}$, $\dfrac{2}{9}$ (*iii*) $2(x+5)$; 10, 18.

4. (*i*) $\dfrac{1}{2\sqrt{x}}$. (*ii*) $-\frac{1}{2}x^{-\frac{3}{2}}$. (*iii*) $b+2cx$.

5. $x/\sqrt{1+x^2}$. 6. $50-32t$, 18, -14ft. per sec.

7. $18-6t$, 6, -6ft. per sec. ; 3 sec.

Ex. IV- Page 29

1. (*i*) $9x^8$. (*ii*) $21x^2$. (*iii*) 0. (*iv*) $\frac{3}{2}x$. (*v*) 0·7.
(*vi*) $3x^9$. (*vii*) $6t$. (*viii*) $-\frac{1}{4}x^{-\frac{3}{2}}$. (*ix*) z^3.
(*x*) $-\dfrac{1}{t^2}$. (*xi*) $-\dfrac{27}{x^{10}}$. (*xii*) $3/2\theta^{\frac{5}{2}}$. (*xiii*) $1/x^{\frac{6}{7}}$.
(*xiv*) $-\dfrac{15}{8x^6}$. (*xv*) 2. (*xvi*) $\dfrac{3}{19}u^{-\frac{16}{19}}$.

3. $\frac{2}{3}kv^{-\frac{1}{3}}$. 4. $4\pi r^2$, 36π, $r=0$ or $\dfrac{1}{4\pi}$. 7. $1{\cdot}4cy^{-2\cdot4}$.

8. (*i*) $6(3x+11)$. (*ii*) $\dfrac{4}{\sqrt{8x+5}}$. (*iii*) $-\frac{5}{2}(5x+3)^{-\frac{3}{2}}$

(*iv*) $\dfrac{1}{(1-x)^2}$. (*v*) $\frac{21}{5}(9-7x)^{-\frac{8}{5}}$.

Ex. V. Page 34.

1. (*i*) x^2+x-1. (*ii*) $2at+b$. (*iii*) $(12-5y)/y^3$.
(*iv*) $(3pz^2+gz-r)/2z^{\frac{3}{2}}$.

2. (*i*) $2x+a+b$. (*ii*) $(y+3)(3y+13)$. (*iii*) $(\theta+1)/\theta^{\frac{3}{2}}$.
(*iv*) $(2z-3)/3z^{\frac{4}{3}}$.

3. $1-\dfrac{1}{2\,!}x^2+\dfrac{1}{4\,!}x^4-\dfrac{1}{6\,!}x^6$.

4. (*i*) 12, $\theta=1$. (*ii*) 166.

7. -2ft. per sec ; 1 or 4 sec. 8. $2\frac{1}{2}$ sec.
9. After $4\frac{1}{2}$ sec. at a distance of 202 ft.

Ex. VI. Page 38.

1. $\frac{1}{5}$. 2. (i) $3(1+x+x^2)^2\ (2x+1)$. (ii) $t/\sqrt{t^2+1}$.
(iii) $5\left(x+\frac{1}{x}\right)^4\left(1-\frac{1}{x^2}\right)$. (iv) $\frac{a-x}{\sqrt{2ax-x^2}}$.
(v) $\frac{t}{2(1-t^2)^{\frac{3}{2}}}$. (vi) $1+\frac{z}{\sqrt{z^2+b^2}}$.
(vii) $7(at+b)\ (at^2+2bt+c)^{\frac{5}{2}}$.
(viii) $\frac{1}{4\sqrt{1+x}\ \sqrt{1+\sqrt{1+x}}}$.

4. $2e\ (a+bc+bet)$. 6. (i) $\frac{1}{t}$. (ii) $2t-1$.
(iii) $\frac{6\theta+1}{15\theta^2}$.

8. (i) $\frac{2-x}{y}$. (ii) $\frac{x(2-3x)}{y(3y-2)}$. (iii) $\sqrt{\frac{a}{x}}$.
(iv) $-\frac{x+3}{y+f}$. (v) $-\frac{b^2x}{a^2y}$.

9. 99. 10. 3f. s. s. 12. $\frac{3}{4}\sqrt{\frac{1+\sqrt{x}}{x}}$.

13. (i) $\frac{3}{4}x^{-\frac{1}{6}}$. (ii) $\frac{2}{5}x^{-3}$.
14. $\frac{(3x^2-3x-5)\ (2x-1)}{2\sqrt{x^2-x}}$.

Ex. VII Page 44.

1. $28x+43$. (ii) $2(3x^2+x+1)$.
(iii) $-84x\ (1+4x)^2\ (1-3x)^3$.
(iv) $5y^4+8y^3-3y^2-10y$.
(v) $(6z-2-5z^2)/\sqrt{3-2z}$.
(vi) $t(1-3t^2)/\sqrt{1-t^2}$.

2. (*i*) $2(2x^3+3x^2-2x-2)$. (*ii*) $5\theta^4-6\theta^2-15$.
(*iii*) $(y+3)(8y^2+51y+69)$.
(*iv*) $10u^4+12u^3+9u^2+14u+7$.

3. (*i*) $\dfrac{x^2-y}{x-y^2}$. (*ii*) $\dfrac{x(x^3-2y^3)}{y(2x^3-y^3)}$. (*iii*) $-\dfrac{ax+hy}{hx+by}$.
(*iv*) $-\dfrac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$.

4. -64.

Ex. VIII Page 47

1. (*i*) $\dfrac{ad-bc}{(cx+d)^2}$. (*ii*) $\dfrac{1-x^2}{(1+x^2)^2}$.
(*iii*) $\dfrac{nx^{n-1}}{(1+x^n)^2}$. (*iv*) $\dfrac{2(x^2-1)}{(1+x+x^2)^2}$.
(*v*) $\dfrac{(x+1)^2(x-5)}{(x-1)^3}$. (*vi*) $\dfrac{-2x}{(1-x)^3}$.
(*vii*) $\dfrac{-2x}{(1+2x)^3}$. (*vii*) $\dfrac{2x^3-3x^4-1}{(1+x-3x^2)^2}$.

2. (*i*) $\dfrac{-1}{(1+x)\sqrt{x+2x}}$. (*ii*) $\dfrac{-a^2}{x^2\sqrt{a^2-x^2}}$.
(*iii*) $\dfrac{a^2}{(a^2-x^2)^{\frac{3}{2}}}$. (*iv*) $\dfrac{x-\sqrt{a^2+x^2}}{a\sqrt{a^2+x^2}}$.
(*v*) $\dfrac{-1}{(1+x)\sqrt{1-x^2}}$. (*vi*) $\dfrac{1}{2(1-2x)\sqrt{x(1-2x)}}$.
(*vii*) $\dfrac{4(x-1)}{(1+x)^3}$. (*viii*) $\dfrac{3(1+2x-x^2)}{(x^2-3)^2(x-3)^2}$.

3. $\dfrac{a}{\sqrt{a^2-x^2}(\sqrt{a^2-x^2}+a)}$.

5. (*i*) $\dfrac{2t}{t^2-1}$. (*ii*) $\dfrac{b(t^2-1)}{2at}$.

6. $2(1+x)^{\frac{1}{2}}\cdot(2+x)^{\frac{3}{2}}$.

7. (*i*) $\dfrac{1}{(1+x^2)^2}$. (*ii*) $\dfrac{(ad-bc)(c'x+d')^2}{(a'd'-b'c')(cx+d)^2}$.

8. $\dfrac{nx^{n+1}-(n+1)x^n+1}{(1-x)^2}$.

9. (*i*) $\dfrac{a}{\sqrt{a^2-x^2}(\sqrt{a^2-x^2}-a)}$. (*ii*) $\dfrac{x}{\sqrt{x^2-1}}+1$.

(*iii*) $2x\left(1+\dfrac{x^2}{\sqrt{x^4-1}}\right)$.

(*iv*) $-\dfrac{2}{x^3}\left(1+\dfrac{1}{\sqrt{1-x^4}}\right)$.

10. $\dfrac{b-2ac}{2(b-cx)\sqrt{(b-cx)(x-2a)}}$.

Ex. IX Page 56

1. (*i*) $-3\sin 3x$. (*ii*) $-n\cos(c-nz)$.

(*iii*) $\dfrac{7\pi}{180}\operatorname{cosec}^2 7x^\circ$. (*iv*) $3\sin 6x$.

(*v*) $-\frac{1}{2}\sin x$. (*vi*) $nx^{n-1}(\sin nx+x\cos nx)$.

(*vii*) $\sec^2\sqrt{y}/\sqrt{y}$. (*viii*) $-2\operatorname{cosec}^4\frac{1}{2}t\cot\frac{1}{2}t$.

(*ix*) $\dfrac{\cos\theta}{2\sqrt{\sin\theta}}$. (*x*) $\dfrac{3\sec^2 z(\tan z-1)}{2\sqrt{\tan z}}$.

2. (*i*) $\frac{1}{2}(5\cos 5z+\cos z)$.
(*ii*) $\cos^4 x-3\cos^2 x\sin^2 x$.

(*iii*) $\sin x(1+\sec^2 x)$. (*iv*) $\dfrac{3(1+3\cos 6x)}{4\sqrt{\cos 3x}}$.

(*v*) $\dfrac{\text{coc}\ \frac{1}{2}\theta-\theta\sin\frac{1}{2}\theta}{2\sqrt{\theta}}$.

(*vi*) $\dfrac{\sec^2\sqrt{x}+\operatorname{cosec}^2\sqrt{x}}{2\sqrt{x}}$.

((*vii*) $\cos 4\theta$. (*viii*) $\sin x$. (*ix*) $-\cos x$.

3. (*i*) $\frac{1+\sin x}{\cos^2 x}$. (*ii*) $\frac{2\sin\theta}{(1+\cos\theta)^2}$.

(*iii*) $\frac{\cos\frac{1}{2}z}{(1-\sin\frac{1}{2}z)^2}$. (*iv*) $2\sec x\,(\sec x+\tan x)^2$.

(*v*) $-\frac{x\sin x}{(1-\cos x)^2}$. (*vi*) $\sec^2 2z$.

(*vii*) $\frac{4x\cos 2x-\sin 2x}{2x\sqrt{x}}$.

(*viii*) $\frac{(z+z^2)\cos 2z-z\sin 2z}{(2+z^2)^2}$.

4. (*a*) (*i*) $m\operatorname{cosec}^2 x\,\sin(m\cot x)$.

(*ii*) $(\sin x+\cos x)\cos(\sin x-\cos x)$

(*b*) $-a(\sin\sqrt{ax+b})\cos\sqrt{ax+b}/2\sqrt{ax+b}$.

5. (*i*) $\frac{\cos x}{2\sqrt{\sin x}}$. (*ii*) $\frac{\cos\sqrt{z}}{2\sqrt{z}}$.

(*iii*) $\frac{\cos\sqrt{x}}{4\sqrt{x\sin\sqrt{x}}}$.

6. (*b*) $4n\tan nx\sec^2 nx$.

7. (*i*) $-\frac{b\cot\phi}{a}$. (*ii*) $\tan\frac{1}{2}\theta$.

(*iii*) $-\frac{b\tan t}{a^2}$.

8. (*i*) $-\cos^2(x+y)-1$.

(*ii*) $\frac{\sin x(2x\cos x-\sin x)}{x^2}$.

(*iii*) $\frac{y\cos x-\sin y}{x\cos y-\sin x}$.

10. (*i*) $\sin^3 x$. (*ii*) $\cos x^2$. (*iii*) $-\sin^2 x/x^2$

Ex. X Page 68

1. (*i*) $\frac{3}{\sqrt{1-9x^2}}$. (*ii*) $-\frac{3}{\sqrt{x-9x}}$.

(*iii*) $-\frac{1}{1+z^2}$. (*iv*) $\frac{2}{\theta\sqrt{\theta^4-1}}$.

(*v*) $\frac{1}{2\sqrt{x}\sqrt{1-x}}$. (*v*) $-\frac{1}{(4x+1)\sqrt{x}}$.

2. (*i*) -1. (*ii*) $2x\tan^{-1}x+1$. (*iii*) $-\frac{2x\sec x^2}{\sqrt{\cos 2x^2}}$

(*iv*) $\frac{1}{1+x^2}$. (*v*) $\sec^{-1}x+\frac{1}{\sqrt{x^2-1}}$.

(*vi*) $\frac{1}{x\sqrt{x^4-1}\sqrt{\sec^{-1}x^2}}$. (*vii*) $\frac{6(\tan^{-1}2x)^2}{1+4x^2}$.

(*viii*) $\frac{1}{5+3\cos x}$.

3. (*i*) $\sin^{-1}x$. (*ii*) $\cos^{-1}2x$. (*iii*) $\sqrt{a^2-x^3}$.

(*iv*) $\frac{\cos x-\sin x}{2(1+\sin x\cos x)}$. (*v*) $\frac{1}{2}$.

(*vi*) $\frac{\cos^{-1}x-x\sqrt{1-x^2}}{(1-x^2)^{\frac{3}{2}}}$.

4. (*i*) $\frac{1}{2}$. (*ii*) $-\frac{1}{2}$.

5. (*i*) $\frac{4}{3\cos x+5}$. (*ii*) $\frac{\sqrt{3}}{2+\cos x}$.

(*iii*) $\frac{-\sqrt{b^2-a^2}}{b+a\cos x}$. (*iv*) $\frac{\sqrt{b^2-a^2}}{b+a\sin x}$.

6. (*i*) $\frac{2}{\sqrt{1-x^2}}$. (*ii*) $\frac{2}{1+x^2}$. (*iii*) $-\frac{2}{1+x^2}$.

(*iv*) $\frac{1}{2(1+x^2)}$.

Ex. XI. Page 72

2. -6. **3.** $(3x+2)(2x+1)^{-\frac{3}{2}}$.

4. (*i*) $\frac{2}{(x+5)^3}$, $\frac{24}{(x+5)^5}$.

(*ii*) $\frac{12}{(x-3)^3}$. $\frac{144}{(x-3)^5}$.

(*iii*) $\frac{2}{(x+1)^3}$, $\frac{24}{(x+1)^5}$.

(*iv*) $-2(25\cos 10x+4\cos 4x)$,
$8(625\cos 10x+16\cos 4x)$.

(*v*) $-2\cos 2x$, $8\cos 2x$.

(*vi*) $\frac{3}{4}(3\sin 3x-\sin x)$, $\frac{3}{4}(\sin x-27\sin 3x)$.

(*vii*) $x^2(12-x^2)\sin x+8x^3\cos x$,
$(x^4-72x^2+24)\sin x+6x(6-x^2)\cos x$.

5. $\frac{3(1+4x^2)}{(1-x^2)^{\frac{7}{2}}}$.

6. (*i*) $120\left[\frac{1}{(x-5)^6}-\frac{1}{(x-6)^6}\right]$.

(*ii*) $512\sin 4x-16\sin 2x$.

9, (*a*) (*i*) 1. (*ii*) 4. (*b*) (*i*) Yes. (*ii*) No.
(*c*) $(0, 1), (-2, -1)$.

R. Q. I. Page 74

1. na^{n-1}.

5. (*i*) $2x\left(1+\frac{x^2}{\sqrt{x^4-1}}\right)$. (*ii*) $2\sec x(\sec x+\tan x)^2$.

iiii) $3\sec^2 3x$.

6. (*a*) (*i*) $-\frac{2xy+y^2}{2xy+x^2}$. (*ii*) $-\frac{2x\sin x^2}{\sqrt{\cos 2x^2}}$.

(*b*) (*i*) $\frac{x\cos\sqrt{1+x^2}}{2\sqrt{1+x^2}\sqrt{\sin\sqrt{1+x^2}}}$.

(*ii*) $\frac{\sqrt{a^2-b^2}}{a+b\cos x}$.

(*c*) (*i*) $\frac{-a}{(a-\sqrt{a^2-x^2})\sqrt{a^2-x^2}}$.

(*ii*) $\frac{1}{2}\sec^2\frac{x}{2}$.

(*iii*) $\frac{2}{1-\sin 2x}$ or $\sec^2\left(\frac{\pi}{4}+x\right)$.

Ex. XII. Page 83.

1. (*i*) 24. (*ii*) $\frac{1}{3}$. (*iii*) -1. (*iv*) $\frac{2}{27}$.
2. $2x$; (*i*) $(2, 4)$; (*ii*) $(-\frac{1}{8}, \frac{1}{64})$.
3. $(1, -9)$, $(-1, -1)$; $(2, -1)$, $(-2, -9)$
4. At 45°. 5. $(-1, -2)$, $(5, 10)$; -8.
6. $(0, a)$, $(a, 0)$; $(a, 0)$, $(-a, 0)$;

$$\left(\frac{1}{2}a, -\frac{\sqrt{3}}{2}a\right), \left(-\frac{1}{2}a, \frac{\sqrt{3}}{2}a\right)$$

7. (*i*) $(\frac{3}{10}, \frac{4}{15})$, $(-\frac{3}{10}, -\frac{4}{15})$

(*ii*) $\left(\frac{3}{2\sqrt{10}}, -\frac{1}{3\sqrt{10}}\right)$, $\left(-\frac{3}{2\sqrt{10}}, \frac{1}{3\sqrt{10}}\right)$

Ex. XIII. Page 88.

1. (*i*) $x-2y=5$; $2x+y=0$.
(*ii*) $2y+x=2$; $y=2x+6$.
(*iii*) $5x-3y=16$; $3x+5y=30$.
(*iv*) $9x+16y=25$; $16x-9y=7$
(*v*) $3x-2y=12$; $2x+3y+5=0$.
(*vi*) $3y+2x=1$; $2y-3x=5$.
(*vii*) $2y+3x=4$, $2x-3y+6=0$.
(*viii*) $y=0$; $x+1=0$.
(*ix*) $8x+9y=16$; $9x-8y=18$.
(*x*) $x+y=\frac{1}{2}\pi$; $x-y=\frac{1}{2}\pi$.
2. $6y=5x+3$; $2y-7x+15=0$.
3. (*i*) $2y=2x+1$. (*ii*) $x+y=\pm\sqrt{2}$.
(*iii*) $2x+y=1$; $2x+y+7=0$.
(*iv*) $2y+3x=4$; $2y+3x+4=0$.
4. $x=1$, $4x+2y=3$.
5. (*i*) $y+8x=2$; $x=8y+49$.
(*ii*) $y+x=3$; $y=x+1$.
(*iii*) $y=\pm\frac{3\sqrt{3}}{8}x-\frac{a}{8}$; $y=\mp\frac{8\sqrt{3}}{9}x+\frac{41a}{36}$

6. (*i*) $xx'+yy'=5$. (*ii*) $yy'+x+x'=0$.
(*iii*) $xx'-yy'=16$. (*iv*) $9xx'+16yy'=25$.
(*v*) $xy'+x'y+12=0$. (*vi*) $3y'^2y=2xx'+x'^2$.
(*vii*) $8y=3\,(x'^2-4)\,x-2x'^3+16$.
(*viii*) $y-y'=6x'\,(x'+1)\,(x-x')$.
(*ix*) $4xx'+9(y'+1)y+9y'=16$.
(*x*) $y-y'=(x-x')\cos 2x'$.

7. $y=2x,\ y+2x=4$. 8. $y=\pm 4,\ x=\pm 4\sqrt{3}$.

9. $3y+8x=10,\ 3y-2x=8$.

Ex. XIV. Page 93.

1. $ty=x+at^2,\ y=-tx+2at+at^3$.

2. $2y-3tx+at^3=0$. 3. $y+1=0$.

4. $y=(2t-1)x+(t-1)^2$; $y=x$; $4y=-1$.

5. $ax\sec\theta-by\operatorname{cosec}\theta=a^2-b^2$.

6. $x-y\cot\frac{1}{2}\theta=a\theta,\ y+x\cot\frac{1}{2}\theta=a(2+\theta\cot\frac{1}{2}\theta)$.

8. $\dfrac{t^2}{\sqrt{1+t^2}}$; $\dfrac{6+t^2}{\sqrt{1+t^2}}$.

9. (*i*) $y=2x$. (*ii*) $y=0$. (*iii*) $x=0$.
(*iv*) $x=3y$, (*v*) $y=x,\ y=-x$.
(*vi*) $x=0,\ y=0$. (*vii*) $y=2x,\ y=-2x$.
(*viii*) $y=0,\ x+y=0$.

10. (*i*) The curve does not pass through the origin
(*ii*) There is no real tangent at the origin.

12. $c^4/\sqrt{x^6+y^6}$. 13. $\frac{4}{9}a$.

14. $x^{\frac{1}{3}}\mathrm{Y}+y^{\frac{1}{3}}\mathrm{X}x={}^{\frac{1}{3}}y^{\frac{1}{3}}a^{\frac{2}{3}}$; $(axy)^{\frac{1}{3}}$; $\frac{1}{2}a(axy)^{\frac{1}{3}}$

15. $x\sin t+y\cos t=a\sin t\cos t$;
$x\cos t-y\sin t=a\cos 2t$

Ex. XV. Page 96.

1. (*i*) 0° the circle and the parabola touch each other at the origin.

(*ii*) 90°, $\tan^{-1}\frac{3}{4}=36°52'$. (*iii*) $\tan^{-1}2=63°26'$.
(*iv*) 90°. (*v*) $\tan^{-1}\frac{1}{3}, \tan^{-1}3$.

(*vi*) $\tan^{-1} 2/\sqrt{\sqrt{2}-1}$.

3. 45°. **4.** 45°. **6.** $\frac{1}{l}-\frac{1}{m}=\frac{1}{l'}-\frac{1}{m'}$

Ex. XVI. Page. 100

1. $\frac{9}{10}$; 90. **2.** $\frac{5}{6}, \frac{5}{8}$. **3.** $\frac{4}{5}\sqrt{26}, \frac{4}{5}, 4\sqrt{26}$, 20.
4. (3, 0) (2, 0). **5.** $a/\sqrt{2}$; $b^2/a\sqrt{2}$.
6. $a \sin^2\theta$, $a \tan\theta \sin^2\theta$. **7.** $a \sin t$, $a \sin \tan^2 \frac{1}{2}t$
8. $\frac{1}{2}$. **10.** $(\frac{16}{5}-\frac{9}{5})$, $(-\frac{16}{5}, \frac{9}{5})$.

R. Q. II Page 101.

2. $y=8x-2$; 26° 34′. **3.** $(\frac{1}{3}a, -\frac{2}{9}a)$.
5. $a=\frac{3}{2}$, $b=\frac{11}{2}$, $c=8$.
6. $\tan^{-1}\frac{12}{41}$; $5\frac{1}{3}$, **7.** $y=x+15$.
9. $\tan^{-1}\frac{1756}{1889}$.
12. $4x+3y=14$; $(-\frac{5}{4}, \frac{19}{3})$.
13. $x\sqrt{3}+5y=10\sqrt{3}$; $5x-y\sqrt{3}=8$.
15. $y=x+2a\pm a\sqrt{2}$.

Ex. XVII. Page 108.

1. (*i*) 95 ft. (*ii*) 53 ft./sec. (*iii*) 24 ft./sec^2.
3. (*b*) (*i*) 48ft./sec. moving upwards.
(*ii*) −16ft./sec. moving downwards.
(*c*) 80ft./sec., $2\frac{1}{2}$sec.
4. 12 sq. in. per sec. **5.** $(\frac{4}{9}, \frac{8}{27})$.
6. $2\frac{2}{3}$ft. per sec. **7.** $y=4/x$. (*a*) Shortened.
(*b*) lengthened at the rate of 6 in. per sec.
8. (*a*) 5π cu. in. per sec., 4π sq. in. per sec.
9. $\frac{1}{3}\pi$. **11.** 24π sq. ft. /sec. **12.** 4 ft. per min.
13. $\frac{1}{3}\sqrt{3}$. **14.** 5·4mi./hr. **15.** 2.13 ft. per sec.

Ex. XVIII Page 113.

1. 0·6 sq. in.; 0·01 sq. in.
3. $\cos\theta\ \delta\theta$; $\pi/360$.
5 15·7 cu. cm. ; 3 per cent.
7. 208 ft. nearly ; 1 per cent.
2. 64·24.
4. $0{\cdot}0864\pi$ sq. in.
6. 10 cu. ft.
8. $\frac{10}{27}\pi=1{\cdot}16$ ft.; 0·58 per cent.

9. 2·1 ft.

11. (*i*) $\dfrac{2b(b-c\cos A)}{a^2}$

(*ii*) $\dfrac{bc\sin A\,\delta A}{\sqrt{b^2+c^2-2bc\cos A}}$.

13. 1 per cent.

14. $\delta h=-a^2b\,\delta b/4c^2h$.

15. 10 miles ; $12\frac{1}{2}$ ft.

Ex XIX Page 118.

1. (*i*) Increasing (*ii*) Increasing, stationary, decreasing.

(*iii*) Increasing, stationary, increasing.

(*iv*) Decreasing. decreasing, increasing, increasing.

Ex. XX Page 124

1. (*i*) Max.$=13$ at $x=1$, min.$=-10$ at $x=2$ and point of inflexion at $x=0$.

(*ii*) Min.$=0$ at $x=0$, max.$=5$ at $x=1$, min.$=-27$ at $x=3$.

2. Max.$=6$ at $x=-1$, min.$=2$ at $x=1$, point of inflexion $(0, 4)$.

3. (*i*) Max. $(-1, 8)$ min. $(1, 4)$.

(*ii*) Max. $(0, 12)$. min. $(-1, 11)$ and $(1, 11)$.

(*iii*) Max. $(0, 4)$, min. $(0, -4)$.

(*iv*) Max. $(1, \frac{1}{3})$, min. $(-1, -1)$.

(*v*) Max. $(-2, -9)$, min. $(2, -1)$.

(*vi*) Min. $(\frac{1}{4}, \frac{27}{95})$ Pt. of inflexion $(1, 0)$.

Ex. XXI Page 128

1. (*i*) Max. at $x=1$; min. at $x=3$, Pt. of inflexion at $x=0$.

(*ii*) Max. at $x=-1$; min. at $x=4$.

(*iii*) Max. $=-3$ at $x=1$, min. $=-128$ at $x=6$.

(*iv*) Max. at $x=-1$, min. at $x=7/9$, pt. of inflexion at $x=3$.

(*v*) Max. at $x=-3, 1$; min. at $x=-1, 3$.
[read $50x^3$ instead of $53x^3$ in the question].

(*vi*) Max. at $x=2$, min. at $x=\frac{1}{6}(11\pm\sqrt{13})$; pt. of inflexion at $x=3$.

2. (*i*) Pt. of inflexion at $x=\frac{1}{2}\pi$. (*ii*) Max.$=4$, min.$=3$.
(*iii*) Max. $=\frac{1}{4}$ at $x=\frac{1}{6}\pi$, $\frac{5}{6}\pi$, min.$=0$ at $x=\frac{1}{2}\pi$, $\frac{3}{2}\pi$.

(*iv*) Max.$=\frac{1}{\sqrt{3}}$, min.$=-\frac{1}{\sqrt{3}}$.

(*v*) Max. $=\frac{1}{4}$, min$=-\frac{1}{4}$. (*vi*) Max.$=1$.

(*vii*) Max$=3\sqrt{3}/16$ min.$=\frac{-3\sqrt{3}}{16}$.

pt. of inflexion at $x=-\frac{1}{2}\pi$, $\frac{1}{2}\pi$.

(*viii*) Max.$=1$, $\frac{1}{9}\sqrt{6}$; minima$=-1$, $-\frac{1}{9}\sqrt{6}$.
(*ix*) Max. $=5$; min.$=0$.

(*x*) Max.$=\frac{1}{2\sqrt{2}}$ at $x=\frac{\pi}{4}$, min. $=-\frac{1}{2\sqrt{2}}$ at $x=\frac{5\pi}{4}$

3. $\theta=\frac{1}{4}\pi$. **5.** $a=1$, $b=0$.

Ex. XXII Page 132.

1. (*i*) Max.$=\sqrt{2}$, min.$=-\sqrt{2}$.
(*ii*) Max.$=5$, min$=-5$.
(*iii*) Max.$=a$, min.$=b$ (when $a>b$).

(*iv*) Max.$=\frac{1}{a^2}$ min.$=\frac{1}{b^2}$ (when $b^2>a^2$).

4. Max.$=3$ for $x=1$, min.$=\frac{1}{3}$ for $x=-1$.
5. (*i*) Max. at $x=\sin^{-1}\frac{1}{4}$, min. at $x=\frac{1}{2}(2n+1)\pi$.
(*ii*) Max. at $x=\pi$, min. at $x=0$ (*iii*) None.

(*iv*) Max. $=1$ at $x=\frac{\pi}{4}$

(*v*) Max. for $\theta=\pm\frac{1}{3}\pi$, min. for $\theta=0$, π.

Ex. XXIII Page 136.

1. $\frac{1}{2}$. **2.** 108.
3. Each$=9$. **4.** P bisects AB.
6. 9 ft. by 18 ft. **7.** 250 by 500.
8. 14″ by 14″ by 28″. **10.** Radius 5 ft. Side 10 ft.
11. $h=r=2$ 6ft. nearly. **12.** $h=2r$.
13 Height$=$twice base radius.
14 $h=\frac{1}{3}\sqrt{3}\,s$, $r=\frac{1}{3}\sqrt{6}\,s$. **15** 2.45 in.
16. (*i*) 2.11 in. (*ii*) 8″ × 8″ × 2″.
17. (*i*) Square. (*ii*) The included angle$=90°$.
18. (*i*) Isosceles.
19. Angle of the sector$=2$ radians.

(*x*) $(\sin x)^{\log x}\left[\cot x \log x+\frac{1}{x}\log \sin x\right]$;

(*xi*) $(\log x)^{\sin x}\left[\frac{\sin x}{x \log x}+\cos x \log (\log x)\right]$;

(*xii*) $(\log \cos x-x \tan x)(\cos x)^{x}$;

(*xiii*) $\left(1+\frac{1}{x}\right)^{x}\left[\log\left(1+\frac{1}{x}\right)-\frac{1}{1+x}\right]$

(*xiv*) $(\cos^{-1}x)^{x}\left(\log \cos^{-1}x-\frac{x.}{\sqrt{1-x^2}\,\cos^{-1}x}\right)$.

2. (*i*) $7(1+x^2)^{7x}\left[\frac{2x^2}{1+x^2}+\log(1+x^2)\right]$

(*ii*) $-2(\cot x)^{\frac{1}{x}}\left[\frac{\log \cot x}{x^2}+\frac{1}{x \sin x \cos x}\right]$

(*iii*) $(\tan x)^{\frac{1}{x}}\left[\frac{1}{x \sin x \cos x}-\frac{1}{x^2}\log \tan x\right]$.

(*iv*) $x^{x}(1+\log x)-(\cos x)^{\frac{1}{x}}\left[\frac{\tan x}{x}+\frac{\log \cos x}{x^2}\right]$

(*v*) $x^{\tan x}\left[\frac{\tan x}{x}+\sec^2 x \text{ long } x\right]$

$+a(\tan ax)^{ax}\left[\frac{2ax}{\sin 2ax}+\log \tan ax\right]$

(*vi*) $(\sin x)^{\tan x}\left[1+\sec^2 x \log \sin x)\right]$

$+\tan x)^{\sin x}\left[\sec x+\cos x \log \tan x\right]$

3. $x^{x}\log(ex)+\frac{1}{x^2}(1-\log x)x^{\frac{1}{x}}$

Ex. XXVII—B. Page 165

1. (*i*) $\left(\frac{2m}{2x+a}+\frac{3n}{3x-a}\right)(2x+a)^{m}(3x-a)^{n}$.

(ii) $\frac{2}{4-x^2} \cdot \frac{\sqrt{x+2}}{\sqrt{2-x}}$.

(iii) $-\frac{3(x^2+5x+8)(x+2)^2}{(x+1)^5(x+4)^3}$. (iv) $\frac{30x(5+x)^2}{(3+x)^6}$.

(v) $-\frac{(22x^2+65x+97)(x+1)^{\frac{1}{2}}}{6(x-2)^{\frac{9}{5}}(x+3)^{\frac{8}{5}}}$.

(iv) $\frac{(2-2x+13x^2+11x^3)(x^2+2)^2}{2\sqrt{x}(x+1)^2}$.

2. (i) $(m\cos^2 x+n\sin^2 x)\frac{\sin^{m-1} x}{\cos^{n+1} x}$; (ii) $\frac{1}{2}\sec^2\frac{1}{2}x$;

(iii) $\frac{1}{(\cos x-\sin x)\sqrt{\cos 2x}}$;

(iv) $2^x e^{\sin^{-1} x}\left[\log 2+\frac{1}{\sqrt{1-x^2}}\right]$;

(v) $a^{2x} e^{\sqrt{x^2+1}}\left[2\log a+\frac{x}{\sqrt{x^2+1}}\right]$.

3. (i) $\frac{2^x \cot x}{\sqrt{x}}\left[\log 2-2\operatorname{cosec} 2x-\frac{1}{2x}\right]$.

(ii) $x^x\left[\sin x^\circ \log ex+\frac{\pi}{180}\cos x^\circ\right]$.

4. (i) $\frac{y^2}{1+y^2}$; (ii) $\frac{1}{2y-1}$; (iii) $\frac{y^2\tan x}{y\log\cos x-1}$.

Ex. XXVIII

1. (i) $a^n \frac{p}{q}\left(\frac{p}{q}-1\right)\left(\frac{p}{q}-2\right)$……………

$$\left(\frac{p}{q}-n+1\right)(ax+b)^{\frac{p}{q}-n};$$

(ii) $a^n e^{ax+b}$. (iii) $(5\log a)^n a^{5x}$.

(iv) $2^n e^{2x}-(-2)^n e^{-2x}$. (v) $\frac{5^n(-1)^{n-1}(n-1)!}{(5x+6)^n}$.

(*vi*) $\frac{5(-1)^{n-1}(n-1)\,!}{7(1+x)^n}$. (*vii*) $3^n \cos(3x+4+\frac{1}{2}n\pi)$

2. (*i*) $-2^{n-1}\cos(2x+\frac{1}{2}n\pi)$;
(*ii*) $\frac{3}{4}\cos(x+\frac{1}{2}n\pi)+\frac{3}{4}\cos(3x+\frac{1}{2}n\pi)$;
(*iii*) $\frac{1}{2}\cos(x+\frac{1}{2}n\pi)-\frac{1}{2}.5^n\cos(5x+\frac{1}{2}n\pi)$;
(*iv*) $(2\log 3)^n.3^{2x}$; (*v*) $(\sqrt{2})^n \sin(\sqrt{2}x+\frac{1}{2}n\pi)$;

(*vi*) $(-1)^{n-1}(n-1)\,!\ \left[\frac{1}{x^n}+\frac{1}{(x+a)^n}\right]$

(*vii*) $\frac{(-1)^n\,n\,!}{3}\left[\frac{1}{(x-1)^{x+1}}-\frac{1}{(x+2)^{n+1}}\right.$.

(*viii*) $\frac{(-1)^n\,n\,!}{a-b}\left[\frac{1}{(x-a)^{n+1}}-\frac{1}{(x-b)^{n+1}}\right]$.

(*ix*) $\frac{(-1)^n\,n\,!}{(x+5)^{n+1}}$.

3. (*i*) $2^{3n-1}\sin(8x+\frac{1}{2}n\pi)+2^{n-1}\sin(2x+\frac{1}{2}n\pi)$.

(*ii*) $(-1)^{n-1}(n-1)\,!\left[\frac{1}{(x+2)^n}+\frac{1}{(x+3)^n}\right]$.

(*iii*) $\frac{(-a)^n(n+2)\,!}{2(ax+b)^{n+3}}$. (*iv*) $\frac{2^{n-1}(n+2)\,!}{(3-2x)^{n+3}}$.

4. (*i*) $(\sqrt{2})^n e^x \sin(x+\frac{1}{4}n\pi)$. (*ii*) $2^n e^x \cos(\sqrt{3}x+\frac{1}{3}n\pi)$.

(*iii*) $(26)^{\frac{1}{2}n}\quad e^{5x}\sin(x+n\tan^{-1}\frac{1}{5})$.

Examples Page 175

(*i*) $(x+n)e^x$. (*ii*) $b^{n-2}e^{bx}[b^2x^2+2bnx+n(n-1)]$
(*iii*) $3^{n-4}e^{3x}[81x^4+108nx^3+54n(n-1)x^2$
$+12n(n-1)(n-2)x+n(n-1)(n-2)(n-3)]$.

(*iv*) $x^3\sin\left(x+\frac{n\pi}{2}\right)+3nx^2\sin\left(x+\frac{n-1}{2}\pi\right)$.

$+3n(n-1)x\sin\left(x+\frac{n-2}{2}\pi\right)$

$+n(n-1)(n-2)\sin\left(x+\frac{n-3}{2}\pi\right)$.

(*v*) $\frac{6(-1)^n(n-4)\,!}{x^{n-3}}$.

4. (a) $e^{2x}.x^{-5}(16x^4-32x^3+48x^2-48x+24)$;
(b) (i) $-243x^2 \sin 3x+810\,x \cos 3x+540 \sin 3x$;
(ii) $e^x(41 \sin 2x-38 \cos 2x)$.

R. Q. IV Page 175

1. (i) n. (ii) e. (iii) e^a ; 3. (i) $\cot x$ (ii) $\sqrt{a/x}$.
4. (i) $2 \operatorname{cosec} 2x \cos(\log \tan x)$. (ii) $\log \sin x+x \cot x$,
6. (i) $2 \sec^3 x$. (ii) $\dfrac{4}{e^{2x}-e^{-2x}}$;

(iii) $\sqrt{-1}$. (iv) $\dfrac{2xy-y^2}{2xy-x^2}$.

Exercise XXIX Page 184.

3. (b) $2x+\frac{2}{3}x^3+\frac{2}{5}x^5+\frac{2}{7}x^7+$ ………… ; 0 020001.
8. 371·25 ft. 9. (i) 0·2588. (ii) 0·0523.
(iii) 0·9397. (iv) 0·1763. (v) 1·6487.

Ex. XXX Page 188.

1. (i) $1+\dfrac{x^2}{2\,!}+\dfrac{5x^4}{4\,!}+\dfrac{61x^6}{6\,!}+$ …………

(ii) $-\dfrac{x^2}{2\,!}-\dfrac{2x^4}{4\,!}-\dfrac{16x^6}{6\,!}-$ …………

2. (i) $x-\dfrac{x^2}{2}+\dfrac{x^3}{6}-\dfrac{x^4}{12}+$ ……………

(ii) $x+\dfrac{x^3}{3!}+\dfrac{5x^5}{5\,!}\quad\dfrac{61x^7}{7\,!}+$ ………………

(iii) $\log 2+\frac{1}{2}x+\frac{1}{8}x^2-\frac{1}{192}x^4+$ ………………
(iv) $x-\frac{1}{2}x^2+\frac{2}{3}x^3+$ ……………

3. (i) $x+\frac{1}{2}x^2-\frac{2}{3}x^3+\frac{1}{4}x^4+\frac{1}{5}x^5-\frac{1}{3}x^6+$ …………
(ii) $4x-5x^2+\frac{28}{3}x^3-\frac{41}{2}x^4+\frac{244}{5}x^5-$ …………
(iii) $1+x+x^2+\frac{1}{2}x^3+$ ……………
(iv) $\frac{1}{3}x^2-\frac{7}{90}x^4+$ …………

(v) $x+x^2+\dfrac{2}{3!}\,x^3-\dfrac{2^2}{5!}\,x^5-$ …………

(vi) $\frac{1}{2}+\frac{1}{4}x-\frac{1}{48}\,x^3+$ ……………

4. (i) $\dfrac{3}{2}$. (ii) $\frac{1}{2}$. (iii) $\dfrac{1}{120}$. (iv) $\log\dfrac{a}{b}$.

5. $a=1$, $b=2$.

Ex. XXXI Page 193

3. (*i*) $e^2\left[1+(x-2)+\frac{(x-2)^2}{2\,!}+\frac{(x-2)^3}{3\,!}+\ldots\ldots\right]$

(*ii*) $27+45(x-3)+22(x-3)^2+3(x-3)^3$.

(*iii*) $1+2(x-\frac{1}{4}\pi)+2(x-\frac{1}{4}\pi)^2+\frac{8}{3}(x-\frac{1}{4}\pi)^3+\ldots\ldots\ldots\ldots$

(*iv*) $\cos 2-(x-2)\sin 2-\frac{(x-2)^2}{2\,!}\cos 2$

$$+\frac{(x-2)^3}{3\,!}\sin 2+\ldots\ldots\ldots\ldots$$

(*v*) $\frac{1}{4}\pi+\frac{1}{2}(x-1)-\frac{1}{4}(x-1)^2+\frac{1}{12}(x-1)^3+\ldots\ldots\ldots\ldots$

5. (*i*) $\cos a-x\sin a-\frac{x^2}{2\,!}\cos a+\frac{x^3}{3!}\sin a+\ldots\ldots\ldots$

(*ii*) $\sin a+(x-a)\cos a-\frac{(x-a)^2}{2\,!}\sin a$

$$-\frac{(x-a)^3}{3\,!}\cos a+\ldots\ldots\ldots\ldots$$

Ex. XXXII Page 195.

1. (*i*) $-2{\cdot}539$. (*ii*) $9{\cdot}010006001$. (*iii*) $-2{\cdot}357375$.
2. (*i*) $0{\cdot}50301$. (*ii*) $0{\cdot}49697$. (*iii*) $0{\cdot}86428$.
(*v*) $0{\cdot}03142$.
3. $1+2x+2x^2+\frac{8}{3}x^3+\frac{10}{3}x^4+\ldots\ldots\ldots$; $1{\cdot}03554$.

Ex. XXXIII Page 203.

[**Note.** Add the constant of integration to the answers of indefinite integrals].

1. $\frac{1}{3}(x-1)^3$; $-\frac{2}{3}(1-x)^{\frac{3}{2}}$; $\log(x-1)$; $-\frac{1}{2(x+1)^2}$;

$\frac{1}{3(1-x)^3}$; $-2\sqrt{1-x}$.

2. $\frac{1}{2}\log(2x+1)$; $-\log(2-x)$; $\frac{1}{2(2-x)^2}$;

$-\frac{1}{5(4+5x)}$; $-\frac{2}{15}(4-5x)^{\frac{3}{2}}$

3. $-\frac{4}{3}(4-\frac{1}{2}x)^{\frac{3}{2}}$; $-\frac{1}{5}(3-2x)^{\frac{5}{2}}$; $-\frac{1}{24}(3-4x)^6$;

$\frac{3}{14}(2x+3)^{\frac{7}{3}}$; $\frac{1}{3}(2x+5)^{\frac{3}{2}}$

4. $\frac{1}{5}e^{5x}$; $-\frac{1}{2}e^{2x}$; $\frac{1}{7}e^{7x+3}$; $\frac{1}{a}e^{ax+b}$

5. $\frac{2^x}{\log 2}$; $\frac{b^x}{\log b}$; $\frac{x^{b+1}}{b+1}$; $\frac{c^{2x}}{2\log c}$

6. $\frac{1}{3}\sin 3x$; $\frac{1}{3}\sin 3x-\frac{1}{2}\cos 2x$; $3\tan\frac{1}{3}\theta$;

$\frac{5}{2}\cos\frac{3-2x}{5}$

7. $-\frac{1}{4}\cos 2x$; $\frac{1}{2}\sin 2\theta$; $\cos(4-x)$; $\frac{1}{n}\tan(nx+a)$

8. $-\frac{1}{2}\cot 2x$; $\sec x$; $-\operatorname{cosec} x$; $-\frac{1}{4}\sec(3-4x)$

9. (*i*) $\operatorname{cosec}\theta-\cot\theta$ (*ii*) $\sec x+\tan x$

10. (*i*) $\cot x$ (*ii*) $\tan x$.

Ex. XXXIV Page 207

1. $3+5t$ **2.** (*i*) $y=1+x^2$ (*ii*) $y=3x^{\frac{3}{2}}-4$

3. $43\frac{3}{4}$; 4 ; 0·01 ; $2\frac{1}{4}$ **4.** 2 ; $\frac{1}{2}\sqrt{2}$; $2\frac{8}{15}$; $8\frac{2}{3}$

5. $\frac{1}{20}$; $\frac{1}{2a}$; $\frac{2}{3}$; $1\frac{1}{6}$ **6.** 1 ; 1 ; $2\log\frac{b}{a}$; $\log 3$

7. $\frac{1}{2}$; 0 ; 1 ; 1 **8.** 0 ; $\frac{1}{2}(\sin 2b-\sin 2a)$; 0.

9. $\sqrt{2}-1$; 0 ; $\log 2$ **10.** $\frac{1}{4}\pi$; $\frac{1}{2}\pi$; $\frac{1}{2}\pi$; $\frac{1}{4}\pi$

Ex. XXXV Page 212

1. (*i*) $\frac{1}{4}x^4-\frac{4}{3}x^3+x^2+x$ (*ii*) $x^3+x^{-2}-6x^{-\frac{2}{3}}$

(*iii*) $\log t+\frac{1}{t}-\frac{1}{t^2}-\frac{1}{3t^3}$ (*iv*) $\frac{2}{3}x\sqrt{x}+2\sqrt{x}$

2. (*i*) $\frac{2}{3}(x+1)^{\frac{3}{2}}-2(x+1)^{\frac{1}{2}}$ (*ii*) $2x+2\log(2x-1)$

(*iii*) $\frac{1}{3}(x+5)^{\frac{3}{2}}+\frac{1}{3}(x+3)^{\frac{3}{2}}$

3. (*i*) $\frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}}-(x+b)^{\frac{3}{2}}\right]$ (*ii*) $\frac{1}{2}x+\frac{1}{4}\sin 2x$

(*iii*) $\frac{1}{12}\cos 3\theta-\frac{3}{4}\cos\theta$ (*iv*) $-x-\cot x$ (*v*) $x+\frac{1}{2}\cos 2x$
(*vi*) $\tan\frac{1}{2}x$ (*vii*) $\tan x-\sec x$ (*viii*) $2\sqrt{2}\sin\frac{1}{2}x$
(*ix*) $\frac{1}{3}\tan 3x-\frac{1}{5}\tan 5x+x$
4. (*i*) $-\frac{1}{4}\cos 2x-\frac{1}{8}\cos 4x$ (*ii*) $-\frac{1}{4}\cos 2x-\frac{1}{16}\cos 4x$

(*iii*) $ax+\dfrac{b^x}{\log b}+\dfrac{c^{2x}}{2\log c}$

5. (*i*) $\frac{1}{6}x^6+\dfrac{5^x}{\log 5}$ (*ii*) $\dfrac{a^x}{\log a}+\dfrac{x^{a+1}}{a+1}$

(*iii*) $2\left[\sqrt{x}-\cos\frac{1}{2}x+e^{\frac{1}{2}x}+\log(x+2)\right]$
6. (*i*) 0 (*ii*) $33\frac{1}{3}$ (*iii*) $4+17\log 2$ (*iv*) 4 (*v*) $\frac{1}{2}\pi c^4$

Ex. XXXVI Page 217

1. $\tan^{-1}x^2$; $\tan^{-1}x^3$ **2.** $-\frac{1}{4}\cos x^4$; $\sin e^x$
3. $\frac{1}{n}\sin x^n$; $\frac{1}{5}\cos^5 x-\frac{1}{3}\cos^3 x$
4. $\sin x-\frac{2}{3}\sin^3 x+\frac{1}{5}\sin^5 x$;
$-\cos x+\cos^3 x-\frac{3}{5}\cos^5 x+\frac{1}{7}\cos^7 x$
5. $-\frac{1}{4}\sqrt{1-4x^2}$; $\frac{1}{2}\sqrt{2x^3-8x+5}$
6. $2\sin\sqrt{x}$; $\log(1+\sin x)$; $\tan^{-1}(\sin\theta)$
7. $\frac{1}{n}\log(x^n+a^n)$; $\frac{1}{2}\log(1+e^{2x})$; $\log(e^x+e^{-x})$
8. $e^{-\frac{1}{x}}$; $\dfrac{1}{e}\log(x^e+e^x)$; $e^x-2\log(e^x-1)+c$.
9. $\log(\log x)$; $\log(\log\sec x)$; $-\cos(\log x)$.
10. $e^{\tan\theta}$; $2\sqrt{\log x}$; $\frac{1}{2}(\log x)^2$; $\log(1+\log x)$.
11. $\frac{1}{2}(\tan^{-1}x)^2$; $\log(\tan^{-1}x)$; $\frac{1}{2}(\sin^{-1}x)^2$
12. $\frac{1}{8}x-\frac{1}{32}\sin 4x$; $\frac{1}{2}\tan^2\theta+\frac{1}{2}\tan^4\theta+\frac{1}{6}\tan^6\theta$;
$\frac{1}{9}\sec^9\theta-\frac{1}{7}\sec^7\theta$.
13. $2\log 2$; $\frac{1}{6}\log 3$; $10\frac{2}{3}$. **14.** 1 ; $1-\frac{1}{4}\pi$, $1\frac{1}{3}$; $\frac{1}{4}\pi-\frac{2}{3}$.
15. $2\tan^{-1}2$; $\frac{3}{8\pi}$; 0.
16. The integrand does not exist for $\theta=0$, a point lying between the given limits.

Ex. XXXVII. Page 225

2. $\frac{1}{4}\tan^{-1}\frac{1}{4}y$; $\frac{1}{3}\tan^{-1}\frac{1}{3}x$; $\frac{1}{a}\tan^{-1}\frac{x}{4a}$, $\frac{1}{8}\log\frac{x-4}{x+4}$.

3. $\sin^{-1}x$, $\sin^{-1}\frac{x}{a}$, $\sqrt{2}\sin^{-1}\frac{x}{2a}$, $\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{x\sqrt{a^2-x^2}}{2}$

4. $\frac{4}{3}\tan^{-1}x^3$, $\frac{3}{4}\sin^{-1}x^3$, $\frac{1}{6a^3}\log\frac{a^3+x^3}{a^3-x^3}$.

5. $\frac{1}{2}\sin^{-1}\frac{1}{2}x^2$, $\frac{1}{2}\sec^{-1}(x^2\sqrt{2})$, $\frac{1}{4}\tan^{-1}\frac{1}{2}x^4$.

6. $\frac{1}{\sqrt{2}}\log\left(\frac{x}{2}+\frac{\pi}{8}\right)$

7. $\frac{1}{\sqrt{a^2+b^2}}\log_e\tan\left(\frac{x}{2}+\frac{\phi}{2}\right)$ where $\tan\phi=\frac{b}{a}$

8. (*i*) $\log\tan x$, (*ii*) $2\operatorname{cosec}2x$.

9. (*i*) $-2\sin x+\log(\operatorname{cosec}x-\cot x)$ (*ii*) $\sqrt{1+\sin^2 x}$.

(*iii*) $x+\tan x+2\log(\sec x+\tan x)$.

10. (*i*) $\log\tan x$; (*ii*) $-2\cot 2x$.

11. (*i*) $\frac{1}{2}\tan 2x-\frac{1}{2}\cot 2x+2\log\tan x$.

(*iii*) $-3\log(1+\cos\frac{1}{3}x)$.

12. (*i*) $\frac{x}{a^2\sqrt{a^2-x^2}}$; (*ii*) $\frac{1}{3}x^3-4x+8\tan^{-1}\frac{x}{2}$

(*iii*) $\frac{1}{3}\sin^{-1}\frac{3x-2}{\sqrt{5}}$; (*iv*) $x+\log\frac{x-2}{x+2}$.

(*v*) $\sin^{-1}(2x-3)$.

13. (*i*) $\frac{1}{3}\tan^{-1}\frac{x+1}{3}$; (*ii*) $\tan^{-1}(x-2)$;

(*iii*) $\frac{1}{6}\log\frac{x-5}{x+1}$.

14. (*i*) $\frac{1}{12}\pi$; (*ii*) $\frac{1}{6}\pi$.

(*iii*) $\frac{\pi}{2\sqrt{2}}$ (*iv*) $\frac{2\pi}{3\sqrt{3}}$ (*v*) π (*vi*) $\frac{1}{4}\pi$.

15. (*i*) $\frac{1}{2}\pi-1$ (*ii*) $2-\frac{1}{2}\pi$ (*iii*) $\frac{1}{4}\pi$.

(*iv*) $\frac{1}{8}\pi$ (*v*) $\log 2$.

Ex XXXVIII Page 231.

1. $(x-1)e^x$, $(x^2-2x+2)e^x$, $\dfrac{ax-1}{a^2}e^{ax}$,
$(x^4-4x^3+12x^2-24x-24)e^x$.

2. $x(\log x-1)$, $\frac{1}{2}x^2(\log x-\frac{1}{2})$, $\frac{1}{3}x^3(\log x-\frac{1}{3})$,
$-\dfrac{1}{2x^2}(\log x+\frac{1}{2})$, $\frac{1}{5}x^5(\log x-\frac{1}{5})$.

3. $\sin x-x\cos x$, $-x^2\cos x+2x\sin x+2\cos x$,
$x\tan x+\log\cos x$, $x\tan x+\log\cos x-\frac{1}{2}x^2$.

4. $\frac{1}{3}x\sin x+\frac{1}{9}\cos 3x$, $\frac{1}{8}(\sin 2x-2x\cos 2x)$,
$\dfrac{x^2}{4}+\dfrac{x\sin 2x}{4}+\dfrac{\cos 2x}{8}$, $\dfrac{x\sin nx}{n}+\dfrac{\cos nx}{n^2}$,
$\dfrac{x^2\sin ax}{a}+\dfrac{2x\cos ax}{a^2}-\dfrac{2\sin ax}{a^3}$.

5. $x\tan^{-1}x-\frac{1}{2}\log(1+x^2)$, $\frac{1}{2}(x^2+1)\tan^{-1}x-\frac{1}{2}x$,
$\frac{1}{4}(2x^2-1)\sin^{-1}x+\frac{1}{4}x\sqrt{1-x^2}$.

6. $\frac{1}{2}e^x(\sin x-\cos x)$, $\frac{1}{2}e^x(\sin x+\cos x)$,
$-\frac{1}{13}e^{-2x}(2\sin 3x+3\cos 3x)$, $\dfrac{e^{ax}(a\cos x+\sin x)}{a^2+1}$.

7. $\frac{1}{2}e^x-\frac{1}{10}e^x(\cos 2x+2\sin 2x)$,
$\frac{1}{2}e^x+\frac{1}{10}e^x(\cos 2x+2\sin 2x)$.

8. $\frac{1}{10}e^x(\sin 2x-2\cos 2x)$, $\frac{1}{68}e^x(\sin 4x-4\cos 4x)$.

9. π, π^2-4.

10. In integrating a function, we must add some arbitrary constant. There will be no fallacy in this case if the arbitrary constant is taken as -1 on the right-hand side of the equation.

Ex. XXXIX Page 236

1. $\log\dfrac{x-3}{x-2}$; (*ii*) $\log\dfrac{x-2}{x-1}$; (*iii*) $\dfrac{1}{a-b}\log\dfrac{(x-a)^a}{(x-b)^b}$.

(*iv*) $\frac{1}{8}\log\dfrac{x-4}{x+4}$; (*v*) $\frac{1}{3}\log\dfrac{1+x^3}{1-x^3}$;

(*vi*) $x^2+3x+3\log(x-1)$.

2. (*i*) $\frac{1}{2}x^3+x+\log(x-1)$; (*ii*) $\frac{1}{3}x^3+4\log(x+2)$;

(*iii*) $\frac{1}{2}x^2+\frac{3}{\sqrt{2}}\tan^{-1}\frac{x}{2}$.

3. (*i*) $\frac{1}{10}\log\frac{x-1}{x+9}$; (*ii*) $3\log(x+3)-\log(x+2)$;

(*iii*) $\log(x+1)+\frac{1}{x+1}$; (*iv*) $x-6\log(x+4)$;

(*v*) $2\log\frac{x}{x+2}+\log(\,;x-2)$

(*vi*) $x-\log(x-1)+4(x-2)$;

4. (*i*) $\frac{1}{2}\log(x+1)+\frac{1}{6}\log(x-1)-\frac{2}{3}\log(x+2)$;

(*ii*) $\frac{x(x+1)^2}{(2x+1)^2}$.

5. (*i*) $\frac{1}{4}\log\frac{x-1}{x+1}-\frac{1}{2}\tan^{-1}x$; (*ii*) $\tan^{-1}x-\frac{1}{\sqrt{2}}\tan^{-1}x\sqrt{2}$;

(*iii*) $\frac{1}{a(a^2-b^2)}\tan^{-1}\frac{x}{a}+\frac{1}{b(b^2-a^2)}\tan^{-1}\frac{x}{b}$.

6. (*i*) $\frac{1}{4}\log\frac{x+1}{x-1}-\frac{1}{2(x-1)}$; (*iv*) $\frac{1}{9}\log\frac{x-1}{x+2}-\frac{2}{3(x-1)}$

(*iii*) $\frac{1}{2(1-x^2)}+\frac{1}{4}\log\frac{x+1}{x-1}$.

7. (*i*) $\log\frac{1+\sqrt{x}}{1-\sqrt{x}}$; (*ii*) $\log(2x-5)+\frac{3}{2}\log(2x-3)$

8. (*i*) $\frac{1}{2}\log(x^2+4x+5)-\tan^{-1}(x+2)$.

(*ii*) $2x-\frac{9}{2}\log(x^2+6x+10)+11\tan^{-1}(x+3)$;

(*iii*) $x-\log(x^2+2x+10)-\frac{8}{3}\tan^{-1}\frac{1}{3}(x+1)$.

9. (*i*) $\frac{1}{2}\log\frac{1+\tan\theta}{1-\tan\theta}$; (*ii*) $\frac{1}{4}\log\frac{\tan\theta}{\sqrt{4+\tan^2\theta}}$.

10. (*i*) $\frac{1}{2}\log\frac{8}{2}$; (*ii*) $\log\frac{4}{3}$.

Ex. XL Page 239

1. 138 ft. per sec. ; 296 ft.

2. (*i*) $v=15+12t-3t^2$, $s=15t+6t^2-t^3$;
(*ii*) After 5 sec. at a distance of 100 ft.

3. 102·5. **4.** 138 ft. ; 14·32 ft. per sec. per sec.

5. 8 ft. per sec. ; $26\frac{1}{4}$ ft. **6.** $y=x^3-x^2-3x+1$.

7. It passes the starting point after $1\frac{1}{2}$ sec. with vel. $-7\frac{1}{2}$ and accln. -4 ; again after 4 sec. with vel. 20 and accln. 26. It changes direction first after $\frac{2}{3}$ sec. at a distance $3\frac{19}{27}$, again after 3 sec. at a distance -9.

Ex. XLI Page 244

1. (*i*) $13\frac{1}{2}$ (*ii*) 120. **2.** $8\frac{2}{3}$. **3.** 60. **4.** $\frac{1}{4}$. **5.** $\frac{8}{3}$, $C^{\frac{5}{2}}$.

6. $c^2 \log \dfrac{a}{b}$. **7.** πab. **8.** (*i*) $2a$. (*ii*) $\frac{1}{2}\pi a$.

9. (*i*) $b \log_e 2$, (*ii*) e^a-1, (*iii*) $c(1-e^{-a})$.

Ex. XLII Page 247

1. (*i*) $\frac{1}{2}\pi^2a^2$; (*ii*) $\frac{1}{4}\pi^2a^2$; (*iii*) $(\pi-\frac{1}{4}\pi^2)a^2$;
(*iv*) $2\pi a^3$; (*v*) $\pi c^4 (a-b)/ab$.

2. (*i*) $\frac{4}{3}\pi ab^2$; (*ii*) $\frac{4}{3}\pi a^2b$. **3.** 79·6 **4.** $\frac{8}{3}\pi a^3$.

PUNJAB UNIVERSITY PAPERS

1951 (April) Page 248

1. (*b*) (*i*) $\dfrac{-2x \sec^2 x^2}{\sqrt{1-\tan^2 x^2}}$ (*ii*) $\frac{1}{2}$

2. (*a*) (*i*) $(\sqrt{2})^n \sin(\sqrt{2}x+\frac{1}{2}n\pi)$.
(*ii*) $2^{n-1} \cos(2x+\frac{1}{2}n\pi)$. (*b*) $-2·539$

4. (*i*) $2\sqrt{2} \sin \frac{1}{2}x+c$ (*ii*) $\log(\operatorname{cosec} x-\cot x)+c$.

(*iii*) $-\cot x-x+c$ (*iv*) $\dfrac{1}{2a^2} \tan^{-1}\dfrac{x^2}{a^2}+c$

(*v*) $-x^2 \cos x+2x \sin x+2\cos x+c$

5. (*b*) $\frac{4}{5}a$ **6.** (*b*) $a=1, b=0$

1951 (Sept.) Page 249

1. (b) (i) $a^{2x}.\ e^{\sqrt{x^2+1}}\left(2\log a+\frac{x}{\sqrt{x^2+1}}\right)$. (ii) $\frac{2}{1+x^2}$

2. (a) (i) $\frac{1}{2}\left(\cos\ (x+\frac{1}{2}n\pi)-5^n\cos\ (5x+\frac{1}{2}n\pi)\right)$.

(ii) $\frac{(-1)^{n-1}\,\underline{|n-1}\,a^n}{(ax+b)^n}$

3. (b) (i) $\sqrt{a^2+b^2}$ max., $-\sqrt{a^2+b^2}$ min.
(ii) max.$=0$ at $x=1$, min.$=-118$ at $x=3$.
Point of inflexion at $x=0$.

4. (i) $\log\ (\text{cosec}\ 2x-\cot\ 2x)+c$ (ii) $\frac{1}{2}\sin x^2+c$

(iii) $\frac{1}{2}x-\frac{1}{4}\sin 2x+c$ (iv) $\sec x+c$

(v) $\frac{1}{4}\sin^{-1}\frac{4x}{3}+c$.

6. (a) $\log\sin x+h\cot x-\frac{h^2}{2}\text{cosec}^2 x+\frac{h^3}{3}\frac{\cos x}{\sin^3 x}\ldots\ldots$;
$-\cdot 6635$.

(b) $x+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5}+\ldots\ldots\ldots\ldots\ldots$

1952 (April). Page 250.

1. (b) (i) $x^{\tan x}\left[\frac{1}{x}\tan x+\sec^2 x\log x\right]$

$+a\ (\tan ax)^{ax}\ [2ax\ \text{cosec}\ 2ax+\log\tan ax]$.

(ii) $\frac{\cos\sqrt{x}}{4\sqrt{x}\sqrt{\sin\ \sqrt{x}}}$.

2. (a) (i) $\frac{a^n(-1)^{n-1}(n-1)!}{(ax+b)^n}$,

(ii) $\frac{3^n}{4}\cos\left(3x+\frac{n\pi}{2}\right)+\frac{3}{4}\cos\left(x+\frac{n\pi}{2}\right)$.

3. (b) Max.$=4$ at $x=n\pi$, min.$=3$ at $x=\frac{1}{2}(2n+1)\pi$.

4. (a) $\frac{1}{2a}\log\frac{x-a}{x+a}+c$; $\frac{1}{2}\log(3x^2+4x+7)+c.$

(b) $\frac{1}{2}\tan^2 x+\log\cos x+c$;

$$\frac{1}{a-b}\left[\frac{2}{3}(x+a)^{\frac{3}{2}}-\frac{2}{3}(x+b)^{\frac{3}{2}}\right]+c.$$

(c) $\frac{x}{a^2\sqrt{a^2-x^2}}+c.$

5. (a) $\frac{1}{l}-\frac{1}{m}=\frac{1}{l'}-\frac{1}{m'}.$

6. 9·010006001.

1952 (Sept.) Page 251

1. (b) (i) $\frac{b-2ac}{2(x-2a)^{\frac{1}{2}}(b-cx)^{\frac{3}{2}}}.$

(ii) $-\frac{2ax+cy}{2by+cx}.$

3. (b) 90° at the point (1, 1). (c) $ax\sec\theta-by\operatorname{cosec}\theta=a^2-b^2.$

4. (a) $-\frac{1}{8(2x^2+3)^2}+c$; $\frac{1}{3}\sin^{-1}\frac{3x-2}{\sqrt{5}}+c.$

(b) $\log(\sec x+\tan x)+c$;

$$\frac{1}{3a\sqrt{5}}\tan^{-1}\frac{x\sqrt{5}}{3a}+c.$$

(c) $\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c.$

5. (a) 2 : 1.

6. (a) $\frac{1}{3}x^3+\frac{7}{90}x^4+\ldots\ldots\ldots\ldots$ (b) (i) $\frac{\sqrt{b^2-a^2}}{b+a\cos x}$

(b) (ii) $(\sin x)^{\tan x}\left[1+\sec^2 x\log\sin x\right]$

$$+(\tan x)^{\sin x}\left[\sec x+\cos x\log\tan x\right].$$

1953 (April) Page 252

1. (b) $-a\cos(\sqrt{ax+b})\sin(\sin\sqrt{ax+b})/2\sqrt{ax+b}.$

2. (a) (i) $\dfrac{\cos x-\sin x}{2+\sin 2x}.$ (ii) $\dfrac{(a^2+b^2)\cos x}{(a+b\sin x)(b-a\sin x)}$

3. (a) $a^n\sin(ax+b+\frac{1}{2}n\pi).$

4. (a) $\sqrt{3}x+5y=10\sqrt{3},\ 5x-\sqrt{3}y=8.$

5. (a) (i) $\dfrac{(x+2x^3)\tan^{-1}x+1}{(1+x^2)^{\frac{3}{2}}}\ e^{x^2}.$

(ii) $\dfrac{(a^2+b^2)\cos x}{(b+a\sin x)\left[(b+a\sin x)^2-(a-b\sin x)^2\right]^{\frac{1}{2}}}$

(b) $\tan\frac{1}{2}\theta.$

6. (a) $\dfrac{1}{a}\tan^{-1}\dfrac{x}{a}+c;\quad \frac{1}{2}\log\dfrac{(x+3)^3}{x+1}+c.$

(b) $\log(\sec x+\tan x)+c;\ -\frac{1}{4}\cos^4 x+c.$

(c) $\sin x-x\cos x+c.\quad \dfrac{e^{ax}(a\cos x+\sin x)}{a^2+1}.$

1953 (Sept.) Page 253

I. (a) (i) $na(ax+b)^{n-1}$ (ii) $\sec x\tan x$

(iii) $\dfrac{a}{2\sqrt{ax+b}}$

(b) $-\dfrac{1}{2\sqrt{x}}\cos\sqrt{x}\,\mathrm{cosec}^2(\sin\sqrt{x})$

II. (a) (i) $\dfrac{1}{x}\log_a e$ (ii) $\dfrac{-\sqrt{ab}}{\sqrt{\sin x\cos x}\,(b\sin x+a\cos x)}$

(b) $\dfrac{2y\,\mathrm{cosec}\,2x-\log\tan y}{2x\,\mathrm{cosec}\,2y-\log\tan x}$

III. (a) (i) $\dfrac{(-)^{n-1}\ 1\cdot3.\cdots\cdots(2n-3)}{2^n\ (x+a)^{n-\frac{1}{2}}}$

(ii) $2^{\frac{n}{2}}e^{x}\sin\left(x+\dfrac{n\pi}{4}\right)$

IV. $\tan^{-1}3$.

V. (*a*) (*i*) $\dfrac{(a^2-b^2)\sin x}{(b+a\cos x)(a+b\cos x)}$ (*ii*) $-\dfrac{a}{x\sqrt{a^2-x^2}}$

(*b*) $\dfrac{x(35x^3-22)}{14x-15}$

VI. (*a*) $\dfrac{1}{\sqrt{2}}\log\tan\left(\dfrac{x}{2}+\dfrac{\pi}{8}\right)$; $\frac{1}{3}\sin^{-1}\dfrac{x^3}{3}$.

(*b*) $\dfrac{x}{2}-\dfrac{\sin 2x}{4}$; $\frac{1}{2}\left[3\log(x+1)+\dfrac{1}{x+1}\right]$

(*c*) $\dfrac{x^2}{4}(2\log x-1)$; $e^x(x^2-2x+2)$

PREFACE

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The book, as the name implies, is meant to be a first course on the subject and its contents and their manner of presentation have all been designed from this point of view.

It has been found from experience that generally students can master the process of differentiation in a mechanical way but find it difficult to understand the concept of limits, continuity and infinity on which it is based. An attempt has, therefore, been made to explain and clarify these concepts by means of concrete and numerical examples.

Differentiation and its application to tangents and normals, maxima and minima values of curves, rate measure, approximate calculations, expansions of certain functions, integration as the inverse of differentiation, its application to problems in mechanics, calculation of areas and volumes of revolution of simple curves—all these have been treated in details and profusely illustrated by carefully selected solved examples and by suitable figures and graphs.

Other examples, which are scattered through the book, are also particularly designed to assist the students to understand the fundamental principles of the subject. The ready made examples are taken mostly from questions set at the University examinations.

It is hoped that the book will prove useful to those for whom it is meant.

I shall be thankful to those who may suggest improvements for future editions or even point out errors in this volume which may have escaped my scrutiny.

S. R. Gupta

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PLANE ANALYTICAL GEOMETRY

A Text Book for Intermediate Students of Indian Universities

BY

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